NONLINEAR ANALYSIS OF CONVENTIONAL AND MICROSTRUCTURE DEPENDENT FUNCTIONALLY GRADED BEAMS UNDER THERMO-MECHANICAL LOADS

A Thesis

by

ARCHANA ARBIND

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

August 2012

Major Subject: Civil Engineering

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Approved by:

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ABSTRACT

Nonlinear Analysis of Conventional and Microstructure Dependent Functionally Graded Beams under Thermo-mechanical Loads. (August 2012)
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Chair of Advisory Committee: Dr. J. N. Reddy

Nonlinear finite element models of functionally graded beams with power-law variation of material, accounting for the von-Kármán geometric nonlinearity and temperature dependent material properties as well as microstructure dependent length scale have been developed using the Euler–Bernoulli as well as the first-order and third-order beam theories. To capture the size effect, a modified couple stress theory with one length scale parameter is used. Such theories play crucial role in predicting accurate deflections of micro- and nano-beam structures. A general third order beam theory for microstructure dependent beam has been developed for functionally graded beams for the first time using a modified couple stress theory with the von Kármán nonlinear strain. Finite element models of the three beam theories have been developed. The thermo-mechanical coupling as well as the bending-stretching coupling play significant role in the deflection response. Numerical results are presented to show the effect of nonlinearity, power-law index, microstructural length scale, and boundary conditions on the bending response of beams under thermo-mechanical loads. In general, the effect of microstructural parameter is to stiffen the beam, while shear deformation has the effect of modeling more realistically as a flexible beam.

To my parents

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my advisor Dr. J. N. Reddy for his extraordinary support and guidance throughout my thesis work. I also would like to thank Drs. C. Arson and Muliana for serving on my thesis committee and providing me constructive comments.

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1. INTRODUCTION

Functionally gradient materials (FGM) are a class of materials that have a gradual variation of material properties from one surface to another [1–3]. In fact, most structures (e.g., sea shells, bones, and so on) in nature are functionally graded to withstand natural forces. In general, all the multi-phase materials fall into the category of functionally gradient materials. In the case of beam, plate, and shell structures, the material variation is assumed to be through the thickness of the structure. FGM structures were first proposed as two-constituent thermal barrier structures for applications in space structures, nuclear reactors, and turbine rotors; they can also be used in flywheels, drill bits, and gears where the surface coatings provide fracture toughness while the bulk material provides strength and stiffness. FGMs used in thermal barrier structures are made of a mixture of ceramic and metal. The ceramic constituent of the material provides the high temperature resistance due to its low thermal conductivity. The ductile metal constituent prevents fracture due to high temperature gradients. The gradation in properties of the material reduces thermal stresses, residual stresses, and stress concentration. Delamination problems associated with traditional composite laminates can be avoided by gradually varying the volume fraction of the constituents rather than abruptly changing them. The gradual variation results in a very efficient material tailored to suit the needs. Most multi-functional materials being designed today are manufactured as functionally graded materials.

Thermal shock occurs during reentry of space vehicles, where the temperature changes from 273°C to about 1,100°C in a few minutes, and the advanced gas turbine, wherein a severe temperature transient of a change in temperature of 1,500°C occurs over a time period of 15 s. Plasma facing materials, propulsion system of planes, cutting tools, engine exhaust liners, aerospace skin structures, incinerator linings, thermal barrier coatings of turbine blades, thermal resistant tiles, and directional heat flux materials are all examples where materials have to operate in extremely high temperature transient environments. Therefore, temperature dependency on material properties should be accounted when modeling FGM structures designed for

This thesis follows the style of Computer Methods in Applied Mechanics and Engineering.

high-temperature applications.

Noda [4] presented an extensive review of FGM structures that covers a wide range of topics from thermo-elastic to thermo-inelastic problems. In this paper, he discussed the importance of temperature dependent properties on thermoelastics problems. He further presented analytical methods to handle transient heat conduction problems and indicated the necessity for the optimization of FGM properties. Zhang et al. [5] modeled an isotropic ceramic/metal laminated beam subjected to an abrupt heating condition and demonstrated the influence of thermomechanical coupling on thermal shock response. In the area of thermoelastic formulations, Tanigawa [6] used a layer-wise model to solve a transient heat conduction problem and optimized the material composition to reduce the thermal stress distribution. Tanigawa [7] also compiled a comprehensive review on the thermoelastic analyses of functionally graded materials. In this review, he discussed closed-form analytical solutions for some simple geometries. These solutions, however, were restricted to steady-state conditions. Tanaka et al. [8, 9] formulated a method to design FGM property profiles using sensitivity and optimization methods based on the criterion of reduction of thermal stresses.

A number of other investigations dealing with thermal stresses and deformation had been published in the literature; see, Noda and his coworkers [10, 11], Reddy and Chin [12], Praveen and Reddy [13], Praveen, Chin, and Reddy [14], Cheng and Batra [15], Vel and Batra [16], and Sankar and Tzeng [17], among others. These studies are concerned with the thermo-elastic analysis of beams, plates, and cylinders made of FGMs. Among these studies that concern the thermo-elastic analysis of plates, beams, or cylinders made of FGMs where the material properties have been considered temperature dependent are Noda and Tsuji [10], Praveen and Reddy [13], Praveen, Chin, and Reddy [14], Yang and Shen [18, 19], and Kitipornchai et al. [20], among few others. The work of Praveen and Reddy [13] also considered the von Kármán nonlinearity (also see Reddy [21, 22]) in functionally graded plates.

Thin micro-beams, which has been very commonly used in many application in micro and nano scale device such as biosensor, micro and nano electro-mechanical devices (MEMS and NEMS), shows microstructure dependent size effect [23]. So, many researcher has employed nonlocal continuum theories to develop beam models. Anthoine [24] has used classical couple stress elasticity theory of Koiter [25] for pure bending of a circular cylinder, which includes four material constants (two classical and two additional). Papargyri-Beskou et al. [26] has developed the higher order Bernoulli Euler beam model based on the gradient elasticity theory with surface energy.The nonlocal Euler-Bernoulli beam model by Peddieson et al. [27], the nonlocal Timoshenko beam model by Wang et al. [28], the nonlocal Euler-Bernoulli, Timoshenko, Reddy, and Levinson beam models formulated by Reddy [29, 30] and Reddy and Pang [31] are developed using constitutive equation proposed by Eringen [32]. Beam models using modified couple stress theory proposed by Yang et al. has been developed by Park and Gao [33, 34], Ma, Gao, and Reddy [35–37], and Aghababaei and Reddy[38] for Timoshenko and Reddy beams and Mindlin plates. These models contains only one material length scale parameter. Governing equations for Euler-Bernoulli and Timoshenko beams have been developed by Reddy [39] using von Kármán nonlinear strains for functionally graded beam and an analytical solution has been presented for linear case. Recently, Reddy [40] and Reddy and Kim [41] presented a general third-order theory of functionally graded plates based on modified couple stress theory.

To the best of the author's knowledge, no results have been reported till to date which concern thermo-mechanical analysis of FGM beams with temperature-dependent properties and account for the microstructural length scale and the von Kármán nonlinearity. This very fact motivated the present study. In this study, a functionally graded through-thickness beams with temperature-dependent material properties and the von Kármán nonlinear strains are considered. Modified couple stress theory is used to account for a microstructural length scale parameter. The beam is subjected to mechanical as well as thermal loads. Three different beam theories, namely, the Euler–Bernoulli beam theory, Timoshenko beam theory, and a general third-order beam theory, are considered in the study. This study aims to investigate the effects of the thermal field, the material grading index, microstructural length scale, and the nonlinearity on the displacement and stress fields under various boundary conditions. The material properties modeled as nonlinear functions of the temperature are assumed to vary along the beam height according to the power-law distribution of the constituent materials.

2. MATERIAL GRADATION AND THICKNESS PROFILE

Consider a beam of length L and rectangular cross section. The x-coordinate is taken along the length of the beam through the geometric centroid,z-coordinate along the thickness (the height) of the beam, and the y-coordinate is taken along the width of the beam, as shown in Fig. 2.1 Generally, the variation of a typical

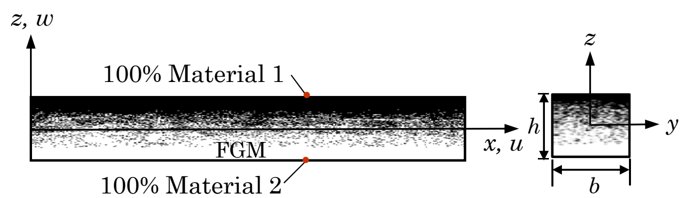


Fig. 2.1 Geometry of a through-thickness functionally graded beam

material property of the material in the FGM beam along the thickness coordinate z is assumed to be represented by the simple power-law as (see Praveen and Reddy [13])

$$P(z,T) = [P_c(T) - P_m(T)]\, f(z) + P_m(T), \quad f(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n \tag{2.1}$$

where P_c and P_m are the material properties of the ceramic and metal faces of the beam, respectively, n is the volume fraction exponent (power-law index). Note that when $n = 0$, we obtain the single-material beam (with property P_c). Fig. 2.2 shows the variation of the volume fraction of ceramic, $f(z)$, through the beam thickness for various values of the power-law index n. Note that the volume fraction $f(z)$ decreases with increasing value of the power-law index n.

Since FGMs are generally used in high temperature environment, the material properties are assumed to be temperature-dependent and expressed in polynomial form as

$$P_\alpha(T) = c_0\left(c_{-1}T^{-1} + 1 + c_1 T + c_2 T^2 + c_3 T^3\right), \quad \alpha = c \text{ or } m \tag{2.2}$$

where c_{-1}, c_1, c_2, and c_3 coefficients of T^{-1}, T, T^2, and T^3, after factoring out c_0 from the cubic curve fit of the property. The material properties were expressed in this way so that the higher-order effects of the temperature on the material properties would be readily discernible. For the analysis with constant properties, the material properties

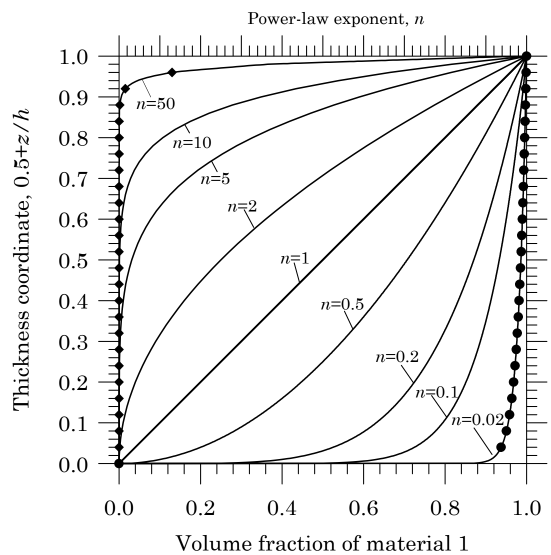


Fig. 2.2 Volume fraction $f(z)$ of the ceramic material through the beam thickness for various values of power-law index n.

were all evaluated at 25.15° C. The values of each of the coefficients appearing in Eq. (2.2) is listed for the metal and ceramic, from Tables 2.1 and 2.2. Also, the material property P at any point along the thickness of the compositionally graded plate is expressed as in Eq. (2.1) The modulus of elasticity, conductivity, and the coefficient of thermal expansion are considered vary according to Eqs. (2.1) and (2.2).

Thermal analysis is carried out by imposing constant surface temperatures at the ceramic and metal rich surfaces. The variation of temperature is assumed to occur in the thickness direction only. The temperature is assumed to be constant along the length of the beam. The thermal analysis is carried out by first solving a simple steady state heat transfer equation through the thickness of the beam, with specified temperature boundary conditions at the top and bottom of the beam. The energy equation for the temperature variation through the thickness is governed by

$$\rho c_v \frac{\partial T}{\partial t} - \frac{\partial}{\partial z}\left[k(z,T)\frac{\partial T}{\partial z}\right] = 0, \quad -\frac{h}{2} \leq z \leq \frac{h}{2} \tag{2.3}$$

$$T(-h/2,t) = T_m(t), \quad T(h/2,t) = T_c(t) \tag{2.4}$$

where $k(z,T)$ is assumed to vary according to Eqs.(2.1) and (2.2), while density ρ and specific heat c_v are assumed to be a constants.

Table 2.1. Material properties of Zirconia

Property	c_0	c_{-1}	$c_1 \times 10^4$	$c_2 \times 10^8$	$c_3 \times 10^{10}$
ρ, Density (kg/m^3)	5700	0	0	0	0
k, Conductivity (W/m K)	1.7	0	1.276	664.85	0
α, Coefficient of thermal expansion (K)	12.7657×10^{-6}	0	-14.9	0.0001	-0.06775
ν, Poisson's ratio	0.2882	0	1.13345	0	0
C_v, Specific heat (J/kg K)	487.34279	0	3.04098	-6.037232	0
E, Young's modulus (Pa)	244.26596×10^9	0	-13.707	121.393	-3.681378

Table 2.2. Material properties of Ti6AIV

Property	c_0	c_{-1}	$c_1 \times 10^4$	$c_2 \times 10^8$	$c_3 \times 10^{10}$
ρ, Density (kg/m^3)	4,429	0	0	0	0
k, Conductivity (W/m K)	1.2094	0	139.375	0	0
α, Coefficient of thermal expansion (K)	7.57876×10^{-6}	0	6.5×10^{-4}	31.467×10^{-8}	0
ν, Poisson's ratio	0.28838235	0	1.12136	0	0
C_v, Specific heat (J/kg K)	625.29692	0	-4.2238757	71.786536	0
E, Young's modulus (Pa)	122.55676 x 10^9	0	-4.58635	0	-3.681378

3. BEAM THEORIES

Equations governing the bending of beams can be formulated using (1) the Euler–Bernoulli beam theory (in which transverse shear strain is assumed to be zero), (2) the Timoshenko beam theory, and (3) the general third-order beam theory (see Reddy [40-44]). In a general beam theory, all applied loads and geometry are such that the total displacements (u_1, u_2, u_3) along the coordinates (x, y, z) are only functions of the x and z coordinates. Here it is further assumed that the displacement u_2 is identically zero.

3.1. Euler-Bernoulli Beam Theory

The total displacements (u_1, u_3) along the coordinate directions (x, z), as implied by the Euler–Bernoulli hypothesis are given by (see Fig. 3.1)

$$u_1(x,z,t) = u(x,t) + z\theta_x(x,t), \quad \theta_x \equiv -\frac{\partial w}{\partial x}, \quad u_3(x,z,t) = w(x,t) \tag{3.1}$$

where w denotes the transverse displacement of a point on the midplane of the beam. The only nonzero von Kármán nonlinear strain is (see Reddy [44])

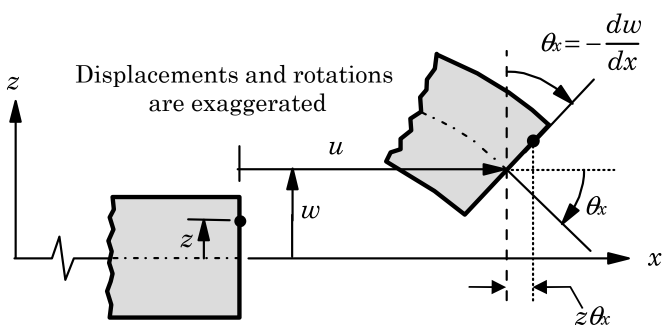


Fig. 3.1 Kinematics of deformation of the Euler–Bernoulli beam theory.

$$\varepsilon_{xx}(x,z,t) = \frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial u_3}{\partial x}\right)^2 = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \theta_x}{\partial x} \tag{3.2}$$

For an isotropic linear elastic material the uniaxial stress-strain relation is used

$$\sigma_{xx}(x,z,t) = E(z,T)\left[\varepsilon_{xx}(x,z,t) - \alpha\Delta T(z,t)\right], \quad \Delta T = T - T_0 \tag{3.3}$$

where E is Young's modulus, α is the coefficient of thermal expansion, and ΔT is the temperature increment from the room temperature, T_0. The temperature T is assumed to vary only through the thickness. The equations of motion associated with the Euler-Bernoulli beam theory are given by (see Reddy [42-45])

$$-\frac{\partial N_{xx}}{\partial x} + I_0\frac{\partial^2 u}{\partial t^2} - I_1\frac{\partial^3 w}{\partial t^2 \partial x} - f \quad = 0 \tag{3.4a}$$

$$\begin{aligned} &-\frac{\partial^2 M_{xx}}{\partial x^2} - \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}\right) \\ &+I_0\frac{\partial^2 w}{\partial t^2} + I_1\frac{\partial^3 u}{\partial t^2 \partial x} - I_2\frac{\partial^4 w}{\partial t^2 \partial x^2} - q \quad = 0 \end{aligned} \tag{3.4b}$$

where $f(x,t)$ and $q(x,t)$ axial and transverse distributed loads on the beam, and (I_0, I_1, I_2) are the mass inertias

$$(I_0, I_1, I_2) = \int_A (1, z, z^2)\rho \, dA \tag{3.5}$$

The stress resultants (N_{xx}, M_{xx}) on a beam element are defined by

$$N_{xx} \quad = \int_A \sigma_{xx}\, dA = A_{xx}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{xx}\frac{\partial \theta_x}{\partial x} - N_{xx}^T \tag{3.6a}$$

$$M_{xx} \quad = \int_A z\sigma_{xx}\, dA = B_{xx}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + D_{xx}\frac{\partial \theta_x}{\partial x} - M_{xx}^T \tag{3.6b}$$

where A_{xx}, B_{xx}, and D_{xx} are the extensional, extensional-bending, and bending stiffness coefficients

$$(A_{xx}(T), B_{xx}(T), D_{xx}(T)) = \int_A (1, z, z^2)E(z,T)\, dA \tag{3.7}$$

and N_{xx}^T and M_{xx}^T are the thermal stress resultants defined by

$$N_{xx}^T(T) = \int_A \alpha(z,T)\, E(z,T)\, \Delta T\, dA, \quad M_{xx}^T(T) = \int_A z\, \alpha(z,T)\, E(z,T)\, \Delta T\, dA \tag{3.8}$$

If we consider the microstructure dependent beam, the equation of motion for Euler–Bernoulli beam can be given as (see Reddy [39]):

$$-\frac{\partial N_{xx}}{\partial x} + I_0\frac{\partial^2 u}{\partial t^2} - I_1\frac{\partial^3 w}{\partial t^2 \partial x} - f \quad = 0 \tag{3.9a}$$

$$-\frac{\partial^2 M_{xx}}{\partial x^2} - \frac{\partial^2 P_{xy}}{\partial x^2} - \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}\right) + I_0\frac{\partial^2 w}{\partial t^2} + I_1\frac{\partial^3 u}{\partial t^2 \partial x} - I_2\frac{\partial^4 w}{\partial t^2 \partial x^2} - q - \frac{\partial r}{\partial x} \quad = 0 \tag{3.9b}$$

where f, q and r are the distributed axial load, transverse load and body couple measured per unit length of the beam.I_0, I_1, I_2 are the mass inertias defined as Eq. (3.5); N_{xx} and M_{xx} have same definition as Eq. (3.6), whereas P_{xy} is defined as

$$P_{xy} = \int_A m_{xy} dA = \frac{1}{2} S_{xy}\left(\frac{\partial \theta_x}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) \tag{3.10}$$

and S_{xy} is defined as

$$S_{xy} = \frac{\ell^2}{2(1+\nu)} \int_A E(z) dA \tag{3.11}$$

here ℓ is microstructure length parameter.

3.2. Timoshenko Beam Theory

The Timoshenko beam theory (TBT) (see Reddy [43]), which is based on the displacement field,

$$u_1(x,z,t) = u(x,t) + z\phi_x(x,t), \quad u_3(x,z,t) = w(x,t) \tag{3.12}$$

where ϕ_x denotes the rotation of the cross section about the y-axis (see Fig. 3.2). In the Timoshenko beam theory the normality assumption of the Euler–Bernoulli beam theory is relaxed and a constant state of transverse shear strain (and thus constant shear stress computed from the constitutive equation) with respect to the thickness coordinate z is included. The Timoshenko beam theory requires shear correction factors to compensate for the error due to this constant shear stress assumption. The shear correction factors depend not only on the material and geometric parameters but also on the load and boundary conditions.

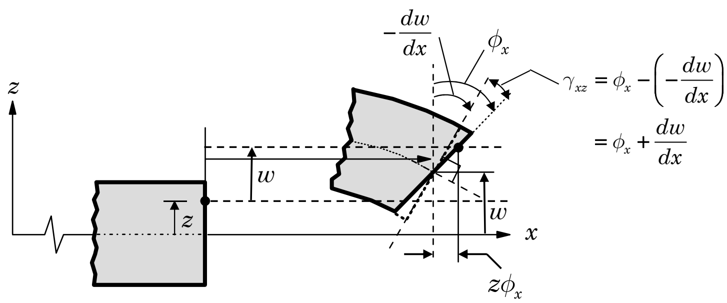


Fig. 3.2 Kinematics of deformation of the Timoshenko beam theory.

The von Kármán nonlinear strains and stresses of the Timoshenko beam theory are

$$\begin{aligned} \varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + z\frac{\partial \phi_x}{\partial x}, \qquad & \gamma_{xz} &= \phi_x + \frac{\partial w}{\partial x} \\ \sigma_{xx} &= E(z,T)\left[\varepsilon_{xx} - \alpha\Delta T\right], & \sigma_{xz} &= G\gamma_{xz} \end{aligned} \tag{3.13}$$

where G the shear modulus $[G = E/2(1+\nu)]$ and ν is Poisson's ratio, which is assumed to be a constant. The equations of motion of the Timoshenko beam theory are

$$-\frac{\partial N_{xx}}{\partial x} + I_0\frac{\partial^2 u}{\partial t^2} + I_1\frac{\partial^2 \phi_x}{\partial t^2} - f = 0 \tag{3.14a}$$

$$-\frac{\partial Q_x}{\partial x} - \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}\right) - q + I_0\frac{\partial^2 w}{\partial t^2} = 0 \tag{3.14b}$$

$$-\frac{\partial M_{xx}}{\partial x} + Q_x + I_2\frac{\partial^2 \phi_x}{\partial t^2} + I_1\frac{\partial^2 u}{\partial t^2} = 0 \tag{3.14c}$$

and boundary conditions (when the corresponding generalized displacements are not specified)

$$\begin{aligned} Q_1 + N_{xx}(0,t) &= 0, \qquad & Q_4 - N_{xx}(L,t) &= 0 \\ Q_2 + V_x(0,t) &= 0, & Q_5 - V_x(L,t) &= 0 \\ Q_3 + M_{xx}(0,t) &= 0, & Q_6 - M_{xx}(L,t) &= 0 \end{aligned} \tag{3.15}$$

where

$$V_x = N_{xx}\frac{\partial w}{\partial x} + Q_x \tag{3.16}$$

The stress resultants are

$$
\begin{aligned}
N_{xx} &= \int_A \sigma_{xx}\, dA = A_{xx}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{xx}\frac{\partial \phi_x}{\partial x} - N_{xx}^T && \text{(3.17a)}\\
M_{xx} &= \int_A z\sigma_{xx}\, dA = B_{xx}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + D_{xx}\frac{\partial \phi_x}{\partial x} - M_{xx}^T && \text{(3.17b)}\\
Q_x &= K_s\int_A \sigma_{xz}\, dA = K_s S_{xz}\left(\phi_x + \frac{\partial w}{\partial x}\right) && \text{(3.17c)}
\end{aligned}
$$

where K_s the shear correction factor and S_{xz} is the shear stiffness

$$
S_{xz} = \frac{1}{2(1+\nu)}\int_A E(z,T)\, dA \tag{3.18}
$$

For microstructure dependent Timoshenko beam, the equation of motion can be given as (see Reddy[39]):

$$
\begin{aligned}
-\frac{\partial N_{xx}}{\partial x} + I_0\frac{\partial^2 u}{\partial t^2} + I_1\frac{\partial^2 \phi_x}{\partial t^2} - f &= 0 && \text{(3.19a)}\\
-\frac{\partial Q_x}{\partial x} - \frac{1}{2}\frac{\partial^2 P_{xy}}{\partial x^2} - \frac{\partial}{\partial x}\left(N_{xx}\frac{\partial w}{\partial x}\right) - q - \frac{1}{2}\frac{\partial r}{\partial x} + I_0\frac{\partial^2 w}{\partial t^2} &= 0 && \text{(3.19b)}\\
-\frac{\partial M_{xx}}{\partial x} + Q_x - \frac{1}{2}\frac{\partial P_{xy}}{\partial x} - \frac{r}{2} + I_2\frac{\partial^2 \phi_x}{\partial t^2} + I_1\frac{\partial^2 u}{\partial t^2} &= 0 && \text{(3.19c)}
\end{aligned}
$$

and the boundary conditions are given by:

$$
\begin{aligned}
&Q_1 + N_{xx}(0,t) = 0, && Q_4 - N_{xx}(L,t) = 0\\
&Q_2 + V_x(0,t) = 0, && Q_5 - V_x(L,t) = 0 \qquad \text{(3.20)}\\
&Q_3 + M_{xx}(0,t) + \frac{1}{2}P_{xy}(0,t) = 0, && Q_6 - M_{xx}(L,t) - \frac{1}{2}P_{xy}(L,t) = 0
\end{aligned}
$$

where N_{xx}, M_{xx}, and Q_x are give by Eq. (3.17) and P_{xy} is given by Eq. (3.10).

3.3. General Third-Order Beam Theory

For general third-order theory, the displacement field for a straight beam bent by forces in the xz plane (i.e., bending about the y-axis) is considered as:

$$
\begin{aligned}
&u_1(x,z,t) = u(x,t) + z\theta_x(x,t) + z^2\phi_x(x,t) + z^3\psi_x(x,t)\\
&u_2(x,z,t) = 0\\
&u_3(x,z,t) = w(x,t) + z\theta_z(x,t) + z^2\phi_z(x,t)
\end{aligned}
\tag{3.21}
$$

where (u, w) are midplane displacements along the x and z directions, respectively, and $(\theta_x, \phi_x, \psi_x, \theta_z, \phi_z)$ have the meaning

$$
\begin{aligned}
&\theta_x = \left(\frac{\partial u_1}{\partial z}\right)_{z=0}, \quad \phi_x = \frac{1}{2}\left(\frac{\partial^2 u_1}{\partial z^2}\right)_{z=0}, \quad \psi_x = \frac{1}{6}\left(\frac{\partial^3 u_1}{\partial z^3}\right)_{z=0}\\
&\theta_z = \left(\frac{\partial u_3}{\partial z}\right)_{z=0}, \quad \phi_z = \frac{1}{2}\left(\frac{\partial^2 u_3}{\partial z^2}\right)_{z=0}
\end{aligned}
\tag{3.22}
$$

The nonzero von Kármán nonlinear strains associated with the displacement field (3.21) are

$$
\begin{aligned}
\varepsilon_{xx} &= \left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + z\frac{\partial \theta_x}{\partial x} + z^2\frac{\partial \phi_x}{\partial x} + z^3\frac{\partial \psi_x}{\partial x}\\
&= \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} + z^2\varepsilon_{xx}^{(2)} + z^3\varepsilon_{xx}^{(3)} && (3.23a)\\
\gamma_{xz} &= \theta_x + \frac{\partial w}{\partial x} + z\left(2\phi_x + \frac{\partial \theta_z}{\partial x}\right) + z^2\left(3\psi_x + \frac{\partial \phi_z}{\partial x}\right)\\
&= \gamma_{xz}^{(0)} + z\gamma_{xz}^{(1)} + z^2\gamma_{xz}^{(2)} && (3.23b)\\
\varepsilon_{zz} &= \theta_z + 2z\phi_z = \varepsilon_{zz}^{(0)} + z\varepsilon_{zz}^{(1)} && (3.23c)\\
\omega_y &= \frac{1}{2}\left[\theta_x - \frac{\partial w}{\partial x} + z\left(2\phi_x - \frac{\partial \theta_z}{\partial x}\right) + z^2\left(3\psi_x - \frac{\partial \phi_z}{\partial x}\right)\right]\\
&= \omega_y^{(0)} + z\omega_y^{(1)} + z^2\omega_y^{(2)} && (3.23d)\\
\chi_{xy} &= 2\chi_{12} = \frac{1}{2}\left[\frac{\partial \theta_x}{\partial x} - \frac{\partial^2 w}{\partial x^2} + z\left(2\frac{\partial \phi_x}{\partial x} - \frac{\partial^2 \theta_z}{\partial x^2}\right) + z^2\left(3\frac{\partial \psi_x}{\partial x} - \frac{\partial^2 \phi_z}{\partial x^2}\right)\right]\\
&= \chi_{xy}^{(0)} + z\chi_{xy}^{(1)} + z^2\chi_{xy}^{(2)} && (3.23e)\\
\chi_{yz} &= 2\chi_{23} = \frac{1}{2}\left[2\phi_x - \frac{\partial \theta_z}{\partial x} + z\left(6\psi_x - 2\frac{\partial \phi_z}{\partial x}\right)\right] = \chi_{yz}^{(0)} + z\chi_{yz}^{(1)} && (3.23f)
\end{aligned}
$$

where

$$\varepsilon_{xx}^{(0)} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_{xx}^{(1)} = \frac{\partial \theta_x}{\partial x}, \quad \varepsilon_{xx}^{(2)} = \frac{\partial \phi_x}{\partial x}, \quad \varepsilon_{xx}^{(3)} = \frac{\partial \psi_x}{\partial x}$$
$$\varepsilon_{zz}^{(0)} = \theta_z, \quad \varepsilon_{zz}^{(1)} = 2\phi_z \tag{3.24}$$
$$\gamma_{xz}^{(0)} = \theta_x + \frac{\partial w}{\partial x}, \quad \gamma_{xz}^{(1)} = \left(2\phi_x + \frac{\partial \theta_z}{\partial x}\right), \quad \gamma_{xz}^{(2)} = \left(3\psi_x + \frac{\partial \phi_z}{\partial x}\right)$$

and

$$\begin{aligned}
\omega_y^{(0)} &= \frac{1}{2}\left(\theta_x - \frac{\partial w}{\partial x}\right), \quad \omega_y^{(1)} = \frac{1}{2}\left(2\phi_x - \frac{\partial \theta_z}{\partial x}\right), \quad \omega_y^{(2)} = \frac{1}{2}\left(3\psi_x - \frac{\partial \phi_z}{\partial x}\right) \\
\chi_{xy}^{(0)} &= \frac{1}{2}\left(\frac{\partial \theta_x}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right), \quad \chi_{xy}^{(1)} = \frac{1}{2}\left(2\frac{\partial \phi_x}{\partial x} - \frac{\partial^2 \theta_z}{\partial x^2}\right), \quad \chi_{xy}^{(2)} = \frac{1}{2}\left(3\frac{\partial \psi_x}{\partial x} - \frac{\partial^2 \phi_z}{\partial x^2}\right) \\
\chi_{yz}^{(0)} &= \frac{1}{2}\left(2\phi_x - \frac{\partial \theta_z}{\partial x}\right), \quad \chi_{yz}^{(1)} = \frac{1}{2}\left(6\psi_x - 2\frac{\partial \phi_z}{\partial x}\right)
\end{aligned} \tag{3.25}$$

Next, Hamilton's Principle has been used to derive the equations of motion incorporating modified couple stress theory and through thickness variation of the material (see Reddy [42–44])

$$\int_{t_2}^{t_1} \left(-\delta K + \delta U + \delta V\right) dt = 0 \tag{3.26}$$

where δK is the virtual kinetic energy, δU is the virtual strain energy, and δV is the virtual work done by external forces. The kinetic energy expression is

$$\begin{aligned}
\delta K &= \int_0^L \int_A \rho \dot{u}_i \delta \dot{u}_i dA dx \\
&= \int_0^L \Big[\left(m_0\dot{u} + m_1\dot{\theta}_x + m_2\dot{\phi}_x + m_3\dot{\psi}_x\right)\delta\dot{u} + \left(m_1\dot{u} + m_2\dot{\theta}_x + m_3\dot{\phi}_x + m_4\dot{\psi}_x\right)\delta\dot{\theta}_x \\
&\quad + \left(m_2\dot{u} + m_3\dot{\theta}_x + m_4\dot{\phi}_x + m_5\dot{\psi}_x\right)\delta\dot{\phi}_x + \left(m_3\dot{u} + m_4\dot{\theta}_x + m_5\dot{\phi}_x + m_6\dot{\psi}_x\right)\delta\dot{\psi}_x \\
&\quad + \left(m_0\dot{w} + m_1\dot{\theta}_z + m_2\dot{\phi}_z\right)\delta w + \left(m_1\dot{w} + m_2\dot{\theta}_z + m_3\dot{\phi}_z\right)\delta\dot{\theta}_z \\
&\quad + \left(m_2\dot{w} + m_3\dot{\theta}_z + m_4\dot{\phi}_z\right)\delta\dot{\phi}_z\Big] dx
\end{aligned} \tag{3.27}$$

where

$$m_i = \int_A \rho\, z^i dA, \quad \text{for} \quad i = 0, 1, 2, 3, 4, 5, 6 \tag{3.28}$$

The expression for the virtual strain energy is

$$
\begin{aligned}
\delta U &= \int_0^L \int_A \left(\sigma_{xx}\,\delta\varepsilon_{xx} + \sigma_{zz}\,\delta\varepsilon_{zz} + \sigma_{xz}\delta\gamma_{xz} + m_{xy}\delta\chi_{xy} + m_{yz}\,\delta\chi_{yz}\right) dAdx \\
&= \int_0^L \Bigg[M_{xx}^{(0)}\left(\frac{\partial\delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial\delta w}{\partial x}\right) + M_{xx}^{(1)}\frac{\partial\delta\theta_x}{\partial x} + M_{xx}^{(2)}\frac{\partial\delta\phi_x}{\partial x} + M_{xx}^{(3)}\frac{\partial\delta\psi_x}{\partial x} \\
&\quad + M_{zz}^{(0)}\delta\theta_z + 2M_{zz}^{(1)}\delta\phi_z + M_{xz}^{(0)}\left(\delta\theta_x + \frac{\partial\delta w}{\partial x}\right) + M_{xz}^{(1)}\left(2\delta\phi_x + \frac{\partial\delta\theta_z}{\partial x}\right) \\
&\quad + M_{xz}^{(2)}\left(3\delta\psi_x + \frac{\partial\delta\phi_z}{\partial x}\right) + \frac{1}{2}\mathcal{M}_{xy}^{(0)}\left(\frac{\partial\delta\theta_x}{\partial x} - \frac{\partial^2\delta w}{\partial x^2}\right) + \frac{1}{2}\mathcal{M}_{xy}^{(1)}\left(2\frac{\partial\delta\phi_x}{\partial x} - \frac{\partial^2\delta\theta_z}{\partial x^2}\right) \\
&\quad + \frac{1}{2}\mathcal{M}_{xy}^{(2)}\left(3\frac{\partial\delta\psi_x}{\partial x} - \frac{\partial^2\delta\phi_z}{\partial x^2}\right) + \frac{1}{2}\mathcal{M}_{yz}^{(0)}\left(2\delta\phi_x - \frac{\partial\delta\theta_z}{\partial x}\right) + \mathcal{M}_{yz}^{(1)}\left(3\delta\psi_x - \frac{\partial\delta\phi_z}{\partial x}\right)\Bigg] dx
\end{aligned}
\tag{3.29}
$$

Various stress resultants used in Eq. (3.29) are defined as

$$
\begin{aligned}
M_{xx}^{(0)} &= \int_A \sigma_{xx} dA, \quad M_{xx}^{(1)} = \int_A z\sigma_{xx}\,dA, \quad M_{xx}^{(2)} = \int_A z^2\sigma_{xx}\,dA, \quad M_{xx}^{(3)} = \int_A z^3\sigma_{xx}\,dA \\
M_{xz}^{(0)} &= \int_A \sigma_{xz} dA, \quad M_{xz}^{(1)} = \int_A z\sigma_{xz}\,dA, \quad M_{xz}^{(2)} = \int_A z^2\sigma_{xz}\,dA, \quad M_{zz}^{(0)} = \int_A \sigma_{zz} dA \\
M_{zz}^{(1)} &= \int_A z\sigma_{zz}\,dA, \quad \mathcal{M}_{xy}^{(0)} = \int_A m_{xy} dA, \quad \mathcal{M}_{xy}^{(1)} = \int_A z m_{xy}\,dA, \quad \mathcal{M}_{xy}^{(2)} = \int_A z^2 m_{xy}\,dA \\
\mathcal{M}_{yz}^{(0)} &= \int_A m_{yz}\,dA, \quad \mathcal{M}_{yz}^{(1)} = \int_A z m_{yz}\,dA
\end{aligned}
\tag{3.30}
$$

The virtual work done by the external forces is

$$
\begin{aligned}
\delta V &= -\int_0^L \Big[\int_A \left(f_x\delta u_1 + f_z\delta u_3 + c_y\delta\omega_y\right) dA + q_x^t\delta u_1(x, \frac{h}{2}) + q_x^b\delta u_1(x, -\frac{h}{2}) \\
&\qquad + q_z^t\delta u_3(x, \frac{h}{2}) + q_z^b\delta u_3(x, -\frac{h}{2})\Big] dx \\
&= -\int_0^L \Big[f_x^{(0)}\delta u + f_x^{(1)}\delta\theta_x + f_x^{(2)}\delta\phi_x + f_x^{(3)}\delta\psi_x + f_z^{(0)}\delta w + f_z^{(1)}\delta\theta_z + f_z^{(2)}\delta\phi_z \\
&\qquad + \frac{1}{2}c_y^{(0)}\left(\delta\theta_x - \frac{\partial\delta w}{\partial x}\right) + \frac{1}{2}c_y^{(1)}\left(2\delta\phi_x - \frac{\partial\delta\theta_z}{\partial x}\right) + \frac{1}{2}c_y^{(2)}\left(3\delta\psi_x - \frac{\partial\delta\phi_z}{\partial x}\right) \\
&\qquad + \left(q_x^t + q_x^b\right)\left(\delta u + \frac{h^2}{4}\delta\phi_x\right) + \frac{h}{2}\left(q_x^t - q_x^b\right)\left(\delta\theta_x + \frac{h^2}{4}\delta\psi_x\right) \\
&\qquad + \left(q_z^t + q_z^b\right)\left(\delta w + \frac{h^2}{4}\delta\phi_z\right) + \frac{h}{2}\left(q_z^t - q_z^b\right)\delta\theta_z\Big] dx
\end{aligned}
\tag{3.31}
$$

where

$$
f_x^{(i)} = \int_A z^i \overline{f}_x\,dA, \quad f_z^{(i)} = \int_A z^i \overline{f}_x\,dA, \quad c_y^{(i)} = \int_A z^i \overline{c}_y\,dA
\tag{3.32}
$$

Here $\overline{f}_x$, $\overline{f}_z$, and $\overline{c}_y$ denote the distributed axial load, transverse load, and body couple about the y axis (all measured per unit volume of the beam), respectively. Substituting δU, δV and δK into the Hamilton's principle (3.26), performing integration-by-parts with respect t as well as x to relieve the generalized displacements δu, $\delta\theta_x$, $\delta\phi_x$, $\delta\psi_x$, δw, $\delta\theta_z$, and $\delta\phi_z$ of any differentiations, and using the fundamental lemma of calculus variations, we obtain the following equations of motion:

$$m_0\frac{\partial^2 u}{\partial t^2}+m_1\frac{\partial^2 \theta_x}{\partial t^2}+m_2\frac{\partial^2 \phi_x}{\partial t^2}+m_3\frac{\partial^2 \psi_x}{\partial t^2}-\frac{\partial M_{xx}^{(0)}}{\partial x}-F_x = 0 \quad (3.33a)$$

$$\begin{aligned} m_1\frac{\partial^2 u}{\partial t^2}+m_2\frac{\partial^2 \theta_x}{\partial t^2}+m_3\frac{\partial^2 \phi_x}{\partial t^2}+m_4\frac{\partial^2 \psi_x}{\partial t^2}-\frac{\partial M_{xx}^{(1)}}{\partial x}+M_{xz}^{(0)}-F_x^{(1)}& \\ -\frac{1}{2}\frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial x}-\frac{1}{2}c_y^{(0)} &= 0 \end{aligned} \quad (3.33b)$$

$$\begin{aligned} m_2\frac{\partial^2 u}{\partial t^2}+m_3\frac{\partial^2 \theta_x}{\partial t^2}+m_4\frac{\partial^2 \phi_x}{\partial t^2}+m_5\frac{\partial^2 \psi_x}{\partial t^2}-\frac{\partial M_{xx}^{(2)}}{\partial x}+2M_{xz}^{(1)}-F_x^{(2)}& \\ -\frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial x}+\mathcal{M}_{yz}^{(0)}-c_y^{(1)} &= 0 \end{aligned} \quad (3.33c)$$

$$\begin{aligned} m_3\frac{\partial^2 u}{\partial t^2}+m_4\frac{\partial^2 \theta_x}{\partial t^2}+m_5\frac{\partial^2 \phi_x}{\partial t^2}+m_6\frac{\partial^2 \psi_x}{\partial t^2}-\frac{\partial M_{xx}^{(3)}}{\partial x}+3M_{xz}^{(2)}-F_x^{(3)}& \\ -\frac{3}{2}\frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial x}+3\mathcal{M}_{yz}^{(1)}-\frac{3}{2}c_y^{(2)} &= 0 \end{aligned} \quad (3.33d)$$

$$\begin{aligned} m_0\frac{\partial^2 w}{\partial t^2}+m_1\frac{\partial^2 \theta_z}{\partial t^2}+m_2\frac{\partial^2 \phi_z}{\partial t^2}-\frac{\partial}{\partial x}\left(M_{xx}^{(0)}\frac{\partial w}{\partial x}\right)-\frac{\partial M_{xz}^{(0)}}{\partial x}-F_z& \\ -\frac{1}{2}\frac{\partial^2 \mathcal{M}_{xy}^{(0)}}{\partial x^2}-\frac{1}{2}\frac{\partial c_y^{(0)}}{\partial x} &= 0 \end{aligned} \quad (3.33e)$$

$$\begin{aligned} m_1\frac{\partial^2 w}{\partial t^2}+m_2\frac{\partial^2 \theta_z}{\partial t^2}+m_3\frac{\partial^2 \phi_z}{\partial t^2}-\frac{\partial M_{xz}^{(1)}}{\partial x}+M_{zz}^{(0)}-F_z^{(1)}& \\ -\frac{1}{2}\frac{\partial^2 \mathcal{M}_{xy}^{(1)}}{\partial x^2}+\frac{1}{2}\frac{\partial \mathcal{M}_{yz}^{(0)}}{\partial x}-\frac{1}{2}\frac{\partial c_y^{(1)}}{\partial x} &= 0 \end{aligned} \quad (3.33f)$$

$$\begin{aligned} m_2\frac{\partial^2 w}{\partial t^2}+m_3\frac{\partial^2 \theta_z}{\partial t^2}+m_4\frac{\partial^2 \phi_z}{\partial t^2}-\frac{\partial M_{xz}^{(2)}}{\partial x}+2M_{zz}^{(1)}-F_z^{(2)}& \\ -\frac{1}{2}\frac{\partial^2 \mathcal{M}_{xy}^{(2)}}{\partial x^2}+\frac{\partial \mathcal{M}_{yz}^{(1)}}{\partial x}-\frac{1}{2}\frac{\partial c_y^{(2)}}{\partial x} &= 0 \end{aligned} \quad (3.33g)$$

where

$$F_x = f_x^{(0)}+q_x^t+q_x^b,\quad F_z = f_z^{(0)}+q_z^t+q_z^b,\quad F_\xi^{(i)} = f_\xi^{(i)}+\left(\frac{h}{2}\right)^i\left[q_\xi^t+(-1)^i q_\xi^b\right]\quad (\xi = x, z) \quad (3.34)$$

The natural boundary conditions involve specifying the following generalized forces:

$$
\begin{aligned}
\delta u : &\qquad M_{xx}^{(0)} \\
\delta\theta_x : &\qquad M_{xx}^{(1)} + \frac{1}{2}\mathcal{M}_{xy}^{(0)} \\
\delta\phi_x : &\qquad M_{xx}^{(2)} + \mathcal{M}_{xy}^{(1)} \\
\delta\psi_x : &\qquad M_{xx}^{(3)} + \frac{3}{2}\mathcal{M}_{xy}^{(2)} \\
\delta w : &\qquad M_{xz}^{(0)} + M_{xx}^{(0)}\frac{\partial w}{\partial x} + \frac{1}{2}\frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial x} + \frac{1}{2}c_y^{(0)} \\
\frac{\partial \delta w}{\partial x} : &\qquad -\frac{1}{2}\mathcal{M}_{xy}^{(0)} \\
\delta\theta_z : &\qquad M_{xz}^{(1)} + \frac{1}{2}\frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial x} - \frac{1}{2}\mathcal{M}_{yz}^{(0)} + \frac{1}{2}c_y^{(1)} \\
\frac{\partial \delta\theta_z}{\partial x} : &\qquad -\frac{1}{2}\mathcal{M}_{xy}^{(1)} \\
\delta\phi_z : &\qquad M_{xz}^{(2)} + \frac{1}{2}\frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial x} - \mathcal{M}_{yz}^{(1)} + \frac{1}{2}c_y^{(2)} \\
\frac{\partial \delta\phi_z}{\partial x} : &\qquad -\frac{1}{2}\mathcal{M}_{xy}^{(2)}
\end{aligned}
\tag{3.35}
$$

4. CONSTITUTIVE EQUATIONS

The constitutive relations for Euler–Bernoulli and Timoshenko beam theories are given in Eqs. (3.3) and (3.13), respectively, assuming isotropic and linear elastic material.

For the general third-order beam theory, 2-D plane stress state is assumed to write the stress-strain relations for isotropic and linear elastic material as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} - \alpha\Delta T \\ \varepsilon_{zz} - \alpha\Delta T \\ \gamma_{xz} \end{Bmatrix} \tag{4.1}$$

$$\begin{aligned} \sigma_{xx} &= \frac{E}{1-\nu^2}\varepsilon_{xx} + \frac{\nu E}{1-\nu^2}\varepsilon_{zz} - \frac{E\alpha}{(1-\nu)}\Delta T \\ \sigma_{zz} &= \frac{\nu E}{1-\nu^2}\varepsilon_{xx} + \frac{E}{1-\nu^2}\varepsilon_{zz} - \frac{E\alpha}{(1-\nu)}\Delta T \\ \sigma_{xz} &= G\gamma_{xz} \end{aligned} \tag{4.2}$$

where E, G, and ν are modulus of elasticity, shear modulus, and Poisson's ratio, respectively. For microstructure dependent beam, the relation between component of couple stress tensor and curvature tensor is taken as (see Reddy [39])

$$m_{xy} = G\ell^2(2\chi_{12}), \quad m_{yz} = G\ell^2(2\chi_{23}) \tag{4.3}$$

where, ℓ is microstructure dependent length factor. Next, the stress resultant for the general third order beam theory can be given as

$$\begin{Bmatrix} \mathcal{M}_{xy}^{(i)} \\ \mathcal{M}_{yz}^{(i)} \end{Bmatrix} = \int_A \begin{Bmatrix} m_{xy} \\ m_{yz} \end{Bmatrix} z^i dA = \sum_{k=i}^{2+i} S_{11}^{(k)} \begin{Bmatrix} \chi_{xy}^{(k-i)} \\ \chi_{yz}^{(k-i)} \end{Bmatrix} \tag{4.4}$$

$$
\begin{aligned}
\left\{ \begin{array}{c} M_{xx}^{(i)} \\ M_{zz}^{(i)} \\ M_{xz}^{(i)} \end{array} \right\} &= \int_A \left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{zz} \\ \sigma_{xz} \end{array} \right\} z^i dA \\
&= \sum_{k=i}^{3+i} \left[\begin{array}{ccc} A_{11}^{(k)} & A_{12}^{(k)} & 0 \\ A_{12}^{(k)} & A_{11}^{(k)} & 0 \\ 0 & 0 & B_{11}^{(k)} \end{array} \right] \left\{ \begin{array}{c} \varepsilon_{xx}^{(k-i)} \\ \varepsilon_{zz}^{(k-i)} \\ \gamma_{xz}^{(k-i)} \end{array} \right\} - \left\{ \begin{array}{c} X_T^{(i)} \\ Z_T^{(i)} \\ 0 \end{array} \right\}
\end{aligned} \tag{4.5}
$$

where

$$
\begin{aligned}
A_{11}^{(k)} &= \frac{1}{1-\nu^2} \int_A z^k\, E(z,T)\, dA, \quad A_{12}^{(k)} = \frac{\nu}{1-\nu^2} \int_A z^k\, E(z,T)\, dA \\
B_{11}^{(k)} &= \frac{1}{2(1+\nu)} \int_A z^k\, E(z,T)\, dA, \quad S_{11}^{(k)} = \frac{\ell^2}{(1+\nu)} \int_A z^k\, E(z,T)\, dA \\
X_T^{(k)} &= Z_T^{(k)} = \frac{1}{(1-\nu)} \int_A z^k\, E(z,T)\, \alpha(z,T) \Delta T dA
\end{aligned} \tag{4.6}
$$

and then the stress resultant in term of various displacements of third order beam theory can be given as:

$$
\begin{aligned}
M_{xx}^{(i)} &= A_{11}^{(i)} \left(\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right) + A_{11}^{(i+1)} \frac{d\theta_x}{dx} + A_{11}^{(i+2)} \frac{d\phi_x}{dx} + A_{11}^{(i+3)} \frac{d\psi_x}{dx} \\
&\quad + A_{12}^{(i)} \theta_z + A_{12}^{(i+1)} 2\phi_z - X_T^{(i)}
\end{aligned} \tag{4.7a}
$$

$$
\begin{aligned}
M_{zz}^{(i)} &= A_{12}^{(i)} \left(\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right) + A_{12}^{(i+1)} \frac{d\theta_x}{dx} + A_{12}^{(i+2)} \frac{d\phi_x}{dx} + A_{12}^{(i+3)} \frac{d\psi_x}{dx} \\
&\quad + A_{11}^{(i)} \theta_z + A_{11}^{(i+1)} 2\phi_z - Z_T^{(i)}
\end{aligned} \tag{4.7b}
$$

$$
M_{xz}^{(i)} = B_{11}^{(i)} \left(\theta_x + \frac{dw}{dx} \right) + B_{11}^{(i+1)} \left(2\phi_x + \frac{d\theta_z}{dx} \right) + B_{11}^{(i+2)} \left(3\psi_x + \frac{d\phi_z}{dx} \right) \tag{4.7c}
$$

$$
\begin{aligned}
\mathcal{M}_{xy}^{(i)} &= \frac{1}{4} S_{11}^{(i)} \left(\frac{d\theta_x}{dx} - \frac{d^2 w}{dx^2} \right) + \frac{1}{4} S_{11}^{(i+1)} \left(2\frac{d\phi_x}{dx} - \frac{d^2 \theta_z}{dx^2} \right) \\
&\quad + \frac{1}{4} S_{11}^{(i+2)} \left(3\frac{d\psi_x}{dx} - \frac{d^2 \phi_z}{dx^2} \right)
\end{aligned} \tag{4.7d}
$$

$$
\mathcal{M}_{yz}^{(i)} = \frac{1}{4} S_{11}^{(i)} \left(2\phi_x - \frac{d\theta_z}{dx} \right) + \frac{1}{4} S_{11}^{(i+1)} \left(6\psi_x - 2\frac{d\phi_z}{dx} \right) \tag{4.7e}
$$

5. VIRTUAL WORK STATEMENTS: WEAK FORMS

5.1. Euler–Bernoulli Beam Theory

5.1.1. Conventional beam

The weak forms of the governing equations of the conventional Euler–Bernoulli beam theory over a typical element $\Omega^e = (x_a, x_b)$ are obtained using the dynamic version of the principle of virtual displacements (see Reddy [42-44]). Let $(\delta u, \delta w)$ be the virtual variations of (u, w). Then the principle of virtual displacements yields the following two weak statements:

$$0 = \int_{x_a}^{x_b} \left[I_0 \delta u \frac{\partial^2 u}{\partial t^2} - I_1 \delta u \frac{\partial^3 w}{\partial t^2 \partial x} + \frac{\partial \delta u}{\partial x} N_{xx} \right] dx - Q_1 \delta u(x_a, t) - Q_4 \delta u(x_b, t) \tag{5.1a}$$

$$\begin{aligned} 0 = \int_{x_a}^{x_b} \Big[& I_0 \delta w \frac{\partial^2 w}{\partial t^2} - I_1 \frac{\partial \delta w}{\partial x} \frac{\partial^2 u}{\partial t^2} + I_2 \frac{\partial \delta w}{\partial x} \frac{\partial^3 w}{\partial x \partial t^2} - \frac{d^2 \delta w}{dx^2} M_{xx} + \frac{\partial \delta w}{\partial x} N_{xx} \frac{\partial w}{\partial x} \\ & - \delta w q \Big] dx - Q_2 \delta w(x_a, t) - Q_3 \delta\theta_x(x_a, t) - Q_5 \delta w(x_b, t) - Q_6 \delta\theta_x(x_b, t) \end{aligned} \tag{5.1b}$$

where Q_i are the generalized forces

$$\begin{aligned} Q_1 &= \left[-N_{xx} \right]_{x_a}, & Q_4 &= \left[N_{xx} \right]_{x_b} \\ Q_2 &= \left[-\frac{\partial M_{xx}}{\partial x} - N_{xx} \frac{\partial w}{\partial x} \right]_{x_a}, & Q_5 &= \left[\frac{\partial M_{xx}}{\partial x} + N_{xx} \frac{\partial w}{\partial x} \right]_{x_b} \\ Q_3 &= \left[-M_{xx} \right]_{x_a}, & Q_6 &= \left[M_{xx} \right]_{x_b} \end{aligned} \tag{5.2}$$

In order to express the weak forms (5.1a) and (5.1b) in terms of the displacements (u, w), N_{xx} and M_{xx} appearing in Eqs. (5.1a) and (5.1b) should be expressed in terms of u and w by means of Eq. (3.4a):

$$\begin{aligned} 0 = \int_{x_a}^{x_b} \Bigg[& I_0 \delta u \frac{\partial^2 u}{\partial t^2} - I_1 \delta u \frac{\partial^3 w}{\partial t^2 \partial x} + \frac{\partial \delta u}{\partial x} \Bigg\{ A_{xx} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] \\ & - B_{xx} \frac{\partial^2 w}{\partial x^2} - N_{xx}^T \Bigg\} \Bigg] dx - \delta u(x_a, t) Q_1 - \delta u(x_b, t) Q_4 \end{aligned} \tag{5.3a}$$

$$
\begin{aligned}
0 \;=\; & \int_{x_a}^{x_b} \Bigg[I_0 \delta w \frac{\partial^2 w}{\partial t^2} - I_1 \frac{\partial \delta w}{\partial x} \frac{\partial^2 u}{\partial t^2} + I_2 \frac{\partial \delta w}{\partial x} \frac{\partial^3 w}{\partial x \partial t^2} + D_{xx} \frac{\partial^2 \delta w}{\partial x^2} \frac{\partial^2 w}{\partial x^2} + M_{xx}^T \frac{\partial^2 \delta w}{\partial x^2} \\
& -B_{xx} \frac{\partial^2 \delta w}{\partial x^2} \left\{ \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right\} + A_{xx} \frac{\partial \delta w}{\partial x} \frac{\partial w}{\partial x} \left\{ \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right\} \\
& -B_{xx} \frac{\partial \delta w}{\partial x} \frac{\partial w}{\partial x} \frac{\partial^2 w}{\partial x^2} - \frac{\partial \delta w}{\partial x} \frac{\partial w}{\partial x} N_{xx}^T - \delta w q \Bigg] dx \\
& -\delta w(x_a,t) Q_2 - \delta w(x_b,t) Q_5 - \delta\theta_x(x_a,t) Q_3 - \delta\theta_x(x_b,t) Q_6 \qquad (5.3b)
\end{aligned}
$$

5.1.2. Microstructure dependent beam

The weak form of Eqs. (3.9a) and (3.9b) for microstructure dependent beam can be given as

$$
0 = \int_{x_a}^{x_b} \left[I_0 \delta u \frac{\partial^2 u}{\partial t^2} - I_1 \delta u \frac{\partial^3 w}{\partial t^2 \partial x} + \frac{\partial \delta u}{\partial x} N_{xx} \right] dx - Q_1 \delta u(x_a,t) - Q_4 \delta u(x_b,t) \qquad (5.4a)
$$

$$
\begin{aligned}
0 \;=\; & \int_{x_a}^{x_b} \Bigg[I_0 \delta w \frac{\partial^2 w}{\partial t^2} - I_1 \frac{\partial \delta w}{\partial x} \frac{\partial^2 u}{\partial t^2} + I_2 \frac{\partial \delta w}{\partial x} \frac{\partial^3 w}{\partial x \partial t^2} - \frac{d^2 \delta w}{d x^2} (M_{xx} + P_{xy}) + \frac{\partial \delta w}{\partial x} N_{xx} \frac{\partial w}{\partial x} \\
& -\delta w q \Bigg] dx - Q_2 \delta w(x_a,t) - Q_3 \delta\theta_x(x_a,t) - Q_5 \delta w(x_b,t) - Q_6 \delta\theta_x(x_b,t) \qquad (5.4b)
\end{aligned}
$$

where Q_i are the generalized forces defined as

$$
\begin{aligned}
& Q_1 = [-N_{xx}]_{x_a}, && Q_4 = [N_{xx}]_{x_b} \\
& Q_2 = \left[-\frac{\partial M_{xx}}{\partial x} - \frac{\partial P_{xy}}{\partial x} - N_{xx} \frac{\partial w}{\partial x} \right]_{x_a}, && Q_5 = \left[\frac{\partial M_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial x} + N_{xx} \frac{\partial w}{\partial x} \right]_{x_b} \qquad (5.5) \\
& Q_3 = [-M_{xx} - P_{xy}]_{x_a}, && Q_6 = [M_{xx} + P_{xy}]_{x_b}
\end{aligned}
$$

By substituting N_{xx}, M_{xx}, and P_{xy} from Eqs. (3.4a) and (3.10), the weak forms (5.4a) and (5.4b) can be expressed in terms of the displacements as:

$$
\begin{aligned}
0 \;=\; & \int_{x_a}^{x_b} \left[I_0 \delta u \frac{\partial^2 u}{\partial t^2} - I_1 \delta u \frac{\partial^3 w}{\partial t^2 \partial x} + \frac{\partial \delta u}{\partial x} \left\{ A_{xx} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] - B_{xx} \frac{\partial^2 w}{\partial x^2} \right. \right. \\
& \left. \left. -N_{xx}^T \right\} \right] dx - \delta u(x_a,t) Q_1 - \delta u(x_b,t) Q_4 \qquad (5.6a)
\end{aligned}
$$

$$
\begin{aligned}
0 = & \int_{x_a}^{x_b} \left\{ I_0 \delta w \frac{\partial^2 w}{\partial t^2} - I_1 \frac{\partial \delta w}{\partial x}\frac{\partial^2 u}{\partial t^2} + I_2 \frac{\partial \delta w}{\partial x}\frac{\partial^3 w}{\partial x \partial t^2} + (D_{xx} + S_{xy}) \frac{\partial^2 \delta w}{\partial x^2}\frac{\partial^2 w}{\partial x^2} \right. \\
& - B_{xx} \frac{\partial^2 \delta w}{\partial x^2} \left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right) + M_{xx}^T \frac{\partial^2 \delta w}{\partial x^2} + A_{xx} \frac{\partial \delta w}{\partial x}\frac{\partial w}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right) \\
& \left. - B_{xx} \frac{\partial \delta w}{\partial x}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2} - \frac{\partial \delta w}{\partial x}\frac{\partial w}{\partial x} N_{xx}^T - \delta w q \right\} dx \\
& - \delta w(x_a,t) Q_2 - \delta w(x_b,t) Q_5 - \delta \theta_x(x_a,t) Q_3 - \delta \theta_x(x_b,t) Q_6 \qquad (5.6b)
\end{aligned}
$$

5.2. Timoshenko Beam Theory

5.2.1. Conventional beam

Let $(\delta u, \delta w, \delta \phi_x)$ be the variations of (u, w, ϕ_x). The principle of virtual displacements applied to the Timoshenko beam theory yields the following three weak forms:

$$
\begin{aligned}
0 = & \int_{x_a}^{x_b} \left[I_0 \delta u \frac{\partial^2 u}{\partial t^2} + I_1 \delta u \frac{\partial^2 \phi_x}{\partial t^2} + \frac{d \delta u}{dx} N_{xx} \right] dx \\
& - Q_1 \delta u(x_a,t) - Q_4 \delta u(x_b,t) \qquad (5.7a) \\
0 = & \int_{x_a}^{x_b} \left[I_0 \delta w \frac{\partial^2 w}{\partial t^2} + \frac{\partial \delta w}{\partial x} Q_x + \frac{\partial \delta w}{\partial x} N_{xx} \frac{\partial w}{\partial x} - \delta w q \right] dx \\
& - Q_2 \delta w(x_a,t) - Q_5 \delta w(x_b,t) \qquad (5.7b) \\
0 = & \int_{x_a}^{x_b} \left[I_1 \delta \phi_x \frac{\partial^2 u}{\partial t^2} + I_2 \delta \phi_x \frac{\partial^2 \phi_x}{\partial t^2} + \delta \phi_x Q_x + \frac{\partial \delta \phi_x}{\partial x} M_{xx} \right] dx \\
& - Q_3 \delta \phi_x(x_a,t) - Q_6 \delta \phi_x(x_b,t) \qquad (5.7c)
\end{aligned}
$$

where Q_i are the generalized forces for the Timoshenko beam theory and can be given as

$$
\begin{aligned}
& Q_1 = \left[-N_{xx}\right]_{x_a}, && Q_4 = \left[N_{xx}\right]_{x_b} \\
& Q_2 = \left[-Q_x - N_{xx}\frac{\partial w}{\partial x}\right]_{x_a}, && Q_5 = \left[Q_x + N_{xx}\frac{\partial w}{\partial x}\right]_{x_b} \qquad (5.8) \\
& Q_3 = \left[-M_{xx}\right]_{x_a}, && Q_6 = \left[M_{xx}\right]_{x_b}
\end{aligned}
$$

The weak forms (5.7a)–(5.7c) can be expressed in terms of the displacements (u, w, ϕ_x) by replacing N_{xx}, M_{xx} and Q_x in terms of u, w, and ϕ_x by means of Eq. (3.17) as

$$
\begin{aligned}
0 &= \int_{x_a}^{x_b} \left[I_0 \delta u \frac{\partial^2 \delta u}{\partial t^2} + I_1 \delta u \frac{\partial^2 \phi_x}{\partial t^2} + \frac{\partial \delta u}{\partial x} \left\{ A_{xx} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + B_{xx} \frac{\partial \phi_x}{\partial x} \right. \right. \\
&\quad \left. \left. - N_{xx}^T \right\} \right] dx - \delta u(x_a, t) Q_1 - \delta u(x_b, t) Q_4 \qquad (5.9a) \\
0 &= \int_{x_a}^{x_b} \left[I_0 \delta w \frac{\partial^2 \delta w}{\partial t^2} + \frac{\partial \delta w}{\partial x} \frac{\partial w}{\partial x} \left\{ A_{xx} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + B_{xx} \frac{\partial \phi_x}{\partial x} - N_{xx}^T \right\} \right. \\
&\quad \left. + K_s S_{xz} \frac{\partial \delta w}{\partial x} \left(\phi_x + \frac{\partial w}{\partial x} \right) - \delta w q \right] dx - Q_2 \delta w(x_a, t) - Q_5 \delta w(x_b, t) \qquad (5.9b) \\
0 &= \int_{x_a}^{x_b} \left[I_1 \delta \phi_x \frac{\partial^2 u}{\partial t^2} + I_2 \delta \phi_x \frac{\partial^2 \delta \phi_x}{\partial t^2} + \frac{\partial \delta \phi_x}{\partial x} \left\{ B_{xx} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + D_{xx} \frac{\partial \phi_x}{\partial x} \right. \right. \\
&\quad \left. \left. - M_{xx}^T \right\} + K_s S_{xz} \delta \phi_x \left(\phi_x + \frac{\partial w}{\partial x} \right) \right] dx - Q_3 \delta \phi_x(x_a, t) - Q_6 \delta \phi_x(x_b, t) \qquad (5.9c)
\end{aligned}
$$

5.2.2. Microstructure dependent beam

For microstructure dependent beam, the weak forms can be given as

$$
\begin{aligned}
0 &= \int_{x_a}^{x_b} \left[I_0 \delta u \frac{\partial^2 u}{\partial t^2} + I_1 \delta u \frac{\partial^2 \phi_x}{\partial t^2} + \frac{d \delta u}{dx} N_{xx} \right] dx - Q_1 \delta u(x_a, t) - Q_5 \delta u(x_b, t) \\
&\qquad (5.10a) \\
0 &= \int_{x_a}^{x_b} \left[I_0 \delta w \frac{\partial^2 w}{\partial t^2} + \frac{\partial \delta w}{\partial x} Q_x + \frac{\partial \delta w}{\partial x} N_{xx} \frac{\partial w}{\partial x} - \frac{1}{2} \frac{\partial^2 \delta w}{\partial x^2} P_{xy} - \delta w (q + \frac{1}{2} \frac{\partial r}{\partial x}) \right] dx \\
&\quad - Q_2 \delta w(x_a, t) - Q_3 \frac{\partial \delta w}{\partial x}(x_a, t) - Q_6 \delta w(x_b, t) - Q_7 \frac{\partial \delta w}{\partial x}(x_b, t) \qquad (5.10b) \\
0 &= \int_{x_a}^{x_b} \left[I_1 \delta \phi_x \frac{\partial^2 u}{\partial t^2} + I_2 \delta \phi_x \frac{\partial^2 \phi_x}{\partial t^2} + \delta \phi_x Q_x + \frac{\partial \delta \phi_x}{\partial x} \left(M_{xx} + \frac{1}{2} P_{xy} \right) \right] dx \\
&\quad - Q_4 \delta \phi_x(x_a, t) - Q_8 \delta \phi_x(x_b, t) \qquad (5.10c)
\end{aligned}
$$

where Q_i are the generalized forces for the Timoshenko beam theory are defined as

$$
\begin{aligned}
&Q_1 = \left[-N_{xx} \right]_{x_a}, && Q_5 = \left[N_{xx} \right]_{x_b} \\
&Q_2 = \left[-Q_x - N_{xx} \frac{\partial w}{\partial x} - \frac{1}{2} \frac{\partial P_{xy}}{\partial x} \right]_{x_a}, && Q_6 = \left[Q_x + N_{xx} \frac{\partial w}{\partial x} + \frac{1}{2} \frac{\partial P_{xy}}{\partial x} \right]_{x_b} \\
&Q_3 = \left[\frac{1}{2} P_{xy} \right]_{x_a}, && Q_7 = \left[-\frac{1}{2} P_{xy} \right]_{x_b} \\
&Q_4 = \left[-M_{xx} - \frac{1}{2} P_{xy} \right]_{x_a}, && Q_8 = \left[M_{xx} + \frac{1}{2} P_{xy} \right]_{x_b}
\end{aligned}
\qquad (5.11)
$$

Again the weak forms (5.10a)–(5.10c) can be expressed in terms of the displacements (u, w, ϕ_x) by replacing N_{xx}, M_{xx},P_{xy} and Q_x in terms of u, w, and ϕ_x by means of Eqs. (3.17) and (3.10):

$$\begin{aligned} 0 &= \int_{x_a}^{x_b} \left[I_0 \delta u \frac{\partial^2 u}{\partial t^2} + I_1 \delta u \frac{\partial^2 \phi_x}{\partial t^2} + \frac{\partial \delta u}{\partial x} \left\{ A_{xx} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + B_{xx} \frac{\partial \phi_x}{\partial x} \right. \right. \\ &\quad \left. \left. - N_{xx}^T \right\} - f \delta u \right] dx - \delta u(x_a, t) Q_1 - \delta u(x_b, t) Q_5 \qquad (5.12a) \\ 0 &= \int_{x_a}^{x_b} \left[I_0 \delta w \frac{\partial^2 w}{\partial t^2} + \frac{\partial \delta w}{\partial x} \frac{\partial w}{\partial x} \left\{ A_{xx} \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + B_{xx} \frac{\partial \phi_x}{\partial x} - N_{xx}^T \right\} \right. \\ &\quad \left. + K_s S_{xz} \frac{\partial \delta w}{\partial x} \left(\phi_x + \frac{\partial w}{\partial x} \right) - \frac{1}{4} \frac{\partial^2 \delta w}{\partial x^2} S_{xy} \left(\frac{\partial \phi_x}{\partial x} - \frac{\partial^2 w}{\partial x^2} \right) - \delta w q \right] dx \\ &\quad - Q_2 \delta w(x_a, t) - Q_3 \frac{\partial \delta w}{\partial x}(x_a, t) - Q_6 \delta w(x_b, t) - Q_7 \frac{\partial \delta w}{\partial x}(x_b, t) \qquad (5.12b) \\ 0 &= \int_{x_a}^{x_b} \left[I_1 \delta \phi_x \frac{\partial^2 u}{\partial t^2} + I_2 \delta \phi_x \frac{\partial^2 \phi_x}{\partial t^2} + \frac{\partial \delta \phi_x}{\partial x} \left\{ B_{xx} \left(\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right) + D_{xx} \frac{\partial \phi_x}{\partial x} \right. \right. \\ &\quad \left. \left. - M_{xx}^T \right\} + K_s S_{xz} \delta \phi_x \left(\phi_x + \frac{\partial w}{\partial x} \right) + \frac{1}{4} \frac{\partial \delta \phi_x}{\partial x} S_{xy} \left(\frac{\partial \phi_x}{\partial x} - \frac{\partial^2 w}{\partial x^2} \right) - \delta \phi_x \frac{r}{2} \right] dx \\ &\quad - Q_4 \delta \phi_x(x_a, t) - Q_8 \delta \phi_x(x_b, t) \qquad (5.12c) \end{aligned}$$

5.3. General Third-Order Beam Theory

Let $(\delta u, \delta w, \delta\theta_x, \delta\phi_x, \delta\psi_x, \delta\theta_z, \delta\phi_z, \delta\psi_z)$ be the variation in $(u, w, \theta_x, \phi_x, \psi_x, \theta_z, \phi_z, \psi_z)$. Then by applying principal of virtual displacement, weak form statements for the general third-order beam theory are

$$
\begin{aligned}
0 &= \int_{x_a}^{x_b} \left[\delta u \left(m_0 \frac{\partial^2 u}{\partial t^2} + m_1 \frac{\partial^2 \theta_x}{\partial t^2} + m_2 \frac{\partial^2 \phi_x}{\partial t^2} + m_3 \frac{\partial^2 \psi_x}{\partial t^2} \right) + \frac{\partial \delta u}{\partial x} M_{xx}^{(0)} - \delta u F_x \right] dx \\
&\quad -\delta u(x_a,t) Q_1 - \delta u(x_b,t) Q_{11} \qquad (5.13a) \\
0 &= \int_{x_a}^{x_b} \left[\delta \theta_x \left(m_1 \frac{\partial^2 u}{\partial t^2} + m_2 \frac{\partial^2 \theta_x}{\partial t^2} + m_3 \frac{\partial^2 \phi_x}{\partial t^2} + m_4 \frac{\partial^2 \psi_x}{\partial t^2} \right) + \frac{\partial \delta \theta_x}{\partial x} M_{xx}^{(1)} + \delta \theta_x M_{xz}^{(0)} \right. \\
&\quad \left. + \frac{1}{2} \frac{\partial \delta \theta_x}{\partial x} \mathcal{M}_{xy}^{(0)} - \delta \theta_x F_x^{(1)} - \frac{1}{2} \delta \theta_x c_y^{(0)} \right] dx - \delta \theta_x(x_a,t) Q_2 - \delta \theta_x(x_b,t) Q_{12} \qquad (5.13b) \\
0 &= \int_{x_a}^{x_b} \left[\delta \phi_x \left(m_2 \frac{\partial^2 u}{\partial t^2} + m_3 \frac{\partial^2 \theta_x}{\partial t^2} + m_4 \frac{\partial^2 \phi_x}{\partial t^2} + m_5 \frac{\partial^2 \psi_x}{\partial t^2} \right) + \frac{\partial \delta \phi_x}{\partial x} M_{xx}^{(2)} \right. \\
&\quad \left. + 2 \delta \phi_x M_{xz}^{(1)} - \delta \phi_x F_x^{(2)} + \frac{\partial \delta \phi_x}{\partial x} \mathcal{M}_{xy}^{(1)} + \delta \phi_x \mathcal{M}_{yz}^{(0)} - \delta \phi_x c_y^{(1)} \right] dx \\
&\quad -\delta \phi_x(x_a,t) Q_3 - \delta \phi_x(x_b,t) Q_{13} \qquad (5.13c) \\
0 &= \int_{x_a}^{x_b} \left[\delta \psi_x \left(m_3 \frac{\partial^2 u}{\partial t^2} + m_4 \frac{\partial^2 \theta_x}{\partial t^2} + m_5 \frac{\partial^2 \phi_x}{\partial t^2} + m_6 \frac{\partial^2 \psi_x}{\partial t^2} \right) + \frac{\partial \delta \psi_x}{\partial x} M_{xx}^{(3)} \right. \\
&\quad \left. + 3 \delta \psi_x M_{xz}^{(2)} - \delta \psi_x F_x^{(3)} + \frac{3}{2} \frac{\partial \delta \psi_x}{\partial x} \mathcal{M}_{xy}^{(2)} + 3 \delta \psi_x \mathcal{M}_{yz}^{(1)} - \frac{3}{2} \delta \psi_x c_y^{(2)} \right] dx \\
&\quad -\delta \psi_x(x_a,t) Q_4 - \delta \psi_x(x_b,t) Q_{14} \qquad (5.13d) \\
0 &= \int_{x_a}^{x_b} \left[\delta w \left(m_0 \frac{\partial^2 w}{\partial t^2} + m_1 \frac{\partial^2 \theta_z}{\partial t^2} + m_2 \frac{\partial^2 \phi_z}{\partial t^2} \right) + \frac{d \delta w}{dx} M_{xx}^{(0)} \frac{\partial w}{\partial x} + \frac{\partial \delta w}{\partial x} M_{xz}^{(0)} \right. \\
&\quad \left. - \delta w F_z - \frac{1}{2} \frac{\partial^2 \delta w}{\partial x^2} \mathcal{M}_{xy}^{(0)} - \delta w \frac{1}{2} \frac{\partial c_y^{(0)}}{\partial x} \right] dx \\
&\quad -\delta w(x_a,t) Q_5 - \frac{\partial \delta w}{\partial x}(x_a,t) Q_6 - \delta w(x_b,t) Q_{15} - \frac{\partial \delta w}{\partial x}(x_b,t) Q_{16} \qquad (5.13e) \\
0 &= \int_{x_a}^{x_b} \left[\delta \theta_z \left(m_1 \frac{\partial^2 w}{\partial t^2} + m_2 \frac{\partial^2 \theta_z}{\partial t^2} + m_3 \frac{\partial^2 \phi_z}{\partial t^2} \right) + \frac{\partial \delta \theta_z}{\partial x} M_{xz}^{(1)} + \delta \theta_z M_{zz}^{(0)} \right. \\
&\quad \left. - \delta \theta_z F_z^{(1)} - \frac{1}{2} \frac{\partial^2 \delta \theta_z}{\partial x^2} \mathcal{M}_{xy}^{(1)} - \frac{1}{2} \frac{\partial \delta \theta_z}{\partial x} \mathcal{M}_{yz}^{(0)} - \frac{1}{2} \delta \theta_z \frac{\partial c_y^{(1)}}{\partial x} \right] dx \\
&\quad -\delta \theta_z(x_a,t) Q_7 - \frac{\partial \delta \theta_z}{\partial x}(x_a,t) Q_8 - \delta \theta_z(x_b,t) Q_{17} - \frac{\partial \delta \theta_z}{\partial x}(x_b,t) Q_{18} \qquad (5.13f) \\
0 &= \int_{x_a}^{x_b} \left[\delta \phi_z \left(m_2 \frac{\partial^2 w}{\partial t^2} + m_3 \frac{\partial^2 \theta_z}{\partial t^2} + m_4 \frac{\partial^2 \phi_z}{\partial t^2} \right) + \frac{\partial \delta \phi_z}{\partial x} M_{xz}^{(2)} + 2 \delta \phi_z M_{zz}^{(1)} \right. \\
&\quad \left. - \delta \phi_z F_z^{(2)} - \frac{1}{2} \frac{\partial^2 \delta \phi_z}{\partial x^2} \mathcal{M}_{xy}^{(2)} - \frac{\partial \delta \phi_z}{\partial x} \mathcal{M}_{yz}^{(1)} - \frac{1}{2} \delta \phi_z \frac{\partial c_y^{(2)}}{\partial x} \right] dx \\
&\quad -\delta \phi_z(x_a,t) Q_9 - \frac{\partial \delta \phi_z}{\partial x}(x_a,t) Q_{10} - \delta \phi_z(x_b,t) Q_{19} - \frac{\partial \delta \phi_z}{\partial x}(x_b,t) Q_{20} \qquad (5.13g)
\end{aligned}
$$

where Q_i are the generalized forces for the general third-order beam theory

$$
\begin{aligned}
Q_1 &= [-M_{xx}^{(0)}]_{x_a}, & Q_{11} &= [M_{xx}^{(0)}]_{x_b}\\
Q_2 &= [-M_{xx}^{(1)} - \frac{1}{2}\mathcal{M}_{xy}^{(0)}]_{x_a}, & Q_{12} &= [M_{xx}^{(1)} + \frac{1}{2}\mathcal{M}_{xy}^{(0)}]_{x_b}\\
Q_3 &= [-M_{xx}^{(2)} - \mathcal{M}_{xy}^{(1)}]_{x_a}, & Q_{13} &= [M_{xx}^{(2)} + \mathcal{M}_{xy}^{(1)}]_{x_b}\\
Q_4 &= [-M_{xx}^{(3)} - \frac{3}{2}\mathcal{M}_{xy}^{(2)}]_{x_a}, & Q_{14} &= [M_{xx}^{(3)} - \frac{3}{2}\mathcal{M}_{xy}^{(2)}]_{x_b}\\
Q_5 &= [-M_{xx}^{(0)}\frac{\partial w}{\partial x} - M_{xz}^{(0)} - \frac{1}{2}\frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial x}]_{x_a}, & Q_{15} &= [M_{xx}^{(0)}\frac{\partial w}{\partial x} + M_{xz}^{(0)} + \frac{1}{2}\frac{\partial \mathcal{M}_{xy}^{(0)}}{\partial x}]_{x_b}\\
Q_6 &= [\frac{1}{2}\mathcal{M}_{xy}^{(0)}]_{x_a}, & Q_{16} &= [-\frac{1}{2}\mathcal{M}_{xy}^{(0)}]_{x_b}\\
Q_7 &= [-M_{xz}^{(1)} - \frac{1}{2}\frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial x} + \frac{1}{2}\mathcal{M}_{yz}^{(0)}]_{x_a}, & Q_{17} &= [M_{xz}^{(1)} + \frac{1}{2}\frac{\partial \mathcal{M}_{xy}^{(1)}}{\partial x} - \frac{1}{2}\mathcal{M}_{yz}^{(0)}]_{x_b}\\
Q_8 &= [\frac{1}{2}\mathcal{M}_{xy}^{(1)}]_{x_a}, & Q_{18} &= [-\frac{1}{2}\mathcal{M}_{xy}^{(1)}]_{x_b}\\
Q_9 &= [-M_{xz}^{(2)} - \frac{1}{2}\frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial x} + \mathcal{M}_{yz}^{(1)}]_{x_a}, & Q_{19} &= [M_{xz}^{(2)} + \frac{1}{2}\frac{\partial \mathcal{M}_{xy}^{(2)}}{\partial x} - \mathcal{M}_{yz}^{(1)}]_{x_b}\\
Q_{10} &= [\frac{1}{2}\mathcal{M}_{xy}^{(2)}]_{x_a}, & Q_{20} &= [-\frac{1}{2}\mathcal{M}_{xy}^{(2)}]_{x_b}
\end{aligned}
\tag{5.14}
$$

The weak form statements (5.13a)-(5.13g) can be given in terms of displacements by substituting various stress resultants from Eq. (4.7) as

$$
\begin{aligned}
0 \;=\; & \int_{x_a}^{x_b} \Bigg[\delta u \left(m_0\frac{\partial^2 u}{\partial t^2} + m_1\frac{\partial^2 \theta_x}{\partial t^2} + m_2\frac{\partial^2 \phi_x}{\partial t^2} + m_3\frac{\partial^2 \psi_x}{\partial t^2}\right)\\
& + \frac{\partial \delta u}{\partial x}\left\{A_{11}^{(0)}\left(\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right) + A_{11}^{(1)}\frac{d\theta_x}{dx} + A_{11}^{(2)}\frac{d\phi_x}{dx} + A_{11}^{(3)}\frac{d\psi_x}{dx}\right.\\
& \left. + A_{12}^{(0)}\theta_z + A_{12}^{(1)}2\phi_z - X_T^{(0)}\right\} - \delta u F_x\Bigg]\,dx - \delta u(x_a,t)Q_1 - \delta u(x_b,t)Q_{11}
\end{aligned}
\tag{5.15a}
$$

$$
\begin{aligned}
0 \;=\; & \int_{x_a}^{x_b} \Bigg[\delta\theta_x \left(m_1\frac{\partial^2 u}{\partial t^2} + m_2\frac{\partial^2 \theta_x}{\partial t^2} + m_3\frac{\partial^2 \phi_x}{\partial t^2} + m_4\frac{\partial^2 \psi_x}{\partial t^2}\right)\\
& + \frac{\partial \delta\theta_x}{\partial x}\left\{A_{11}^{(1)}\left(\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right) + A_{11}^{(2)}\frac{d\theta_x}{dx} + A_{11}^{(3)}\frac{d\phi_x}{dx} + A_{11}^{(4)}\frac{d\psi_x}{dx} + A_{12}^{(1)}\theta_z\right.\\
& \left. + 2A_{12}^{(2)}\phi_z - X_T^{(1)}\right\} + \delta\theta_x\left\{B_{11}^{(0)}\left(\theta_x + \frac{dw}{dx}\right) + B_{11}^{(1)}\left(2\phi_x + \frac{d\theta_z}{dx}\right)\right.\\
& \left. + B_{11}^{(2)}\left(3\psi_x + \frac{d\phi_z}{dx}\right)\right\} + \frac{1}{2}\frac{\partial \delta\theta_x}{\partial x}\left\{\frac{1}{4}S_{11}^{(0)}\left(\frac{d\theta_x}{dx} - \frac{d^2 w}{dx^2}\right) + \frac{1}{4}S_{11}^{(1)}\left(2\frac{d\phi_x}{dx} - \frac{d^2\theta_z}{dx^2}\right)\right.\\
& \left. + \frac{1}{4}S_{11}^{(2)}\left(3\frac{d\psi_x}{dx} - \frac{d^2\phi_z}{dx^2}\right)\right\} - \delta\theta_x F_x^{(1)} - \frac{1}{2}\delta\theta_x c_y^{(0)}\Bigg]\,dx
\end{aligned}
$$

$$-\delta\theta_x(x_a,t)Q_2-\delta\theta_x(x_b,t)Q_{12} \tag{5.15b}$$

$$\begin{aligned}
0 = &\int_{x_a}^{x_b}\left[\delta\phi_x\left(m_2\frac{\partial^2 u}{\partial t^2}+m_3\frac{\partial^2\theta_x}{\partial t^2}+m_4\frac{\partial^2\phi_x}{\partial t^2}+m_5\frac{\partial^2\psi_x}{\partial t^2}\right)\right.\\
&+\frac{\partial\delta\phi_x}{\partial x}\left\{A_{11}^{(2)}\left(\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right)+A_{11}^{(3)}\frac{d\theta_x}{dx}+A_{11}^{(4)}\frac{d\phi_x}{dx}+A_{11}^{(5)}\frac{d\psi_x}{dx}\right.\\
&\left.+A_{12}^{(2)}\theta_z+2A_{12}^{(3)}\phi_z-X_T^{(2)}\right\}+2\delta\phi_x\left\{B_{11}^{(1)}\left(\theta_x+\frac{dw}{dx}\right)+B_{11}^{(2)}\left(2\phi_x+\frac{d\theta_z}{dx}\right)\right.\\
&\left.+B_{11}^{(3)}\left(3\psi_x+\frac{d\phi_z}{dx}\right)\right\}-\delta\phi_x F_x^{(2)}+\frac{\partial\delta\phi_x}{\partial x}\left\{\frac{1}{4}S_{11}^{(1)}\left(\frac{d\theta_x}{dx}-\frac{d^2w}{dx^2}\right)\right.\\
&\left.+\frac{1}{4}S_{11}^{(2)}\left(2\frac{d\phi_x}{dx}-\frac{d^2\theta_z}{dx^2}\right)+\frac{1}{4}S_{11}^{(3)}\left(3\frac{d\psi_x}{dx}-\frac{d^2\phi_z}{dx^2}\right)\right\}\\
&\left.+\delta\phi_x\left\{\frac{1}{4}S_{11}^{(0)}\left(2\phi_x-\frac{d\theta_z}{dx}\right)+\frac{1}{4}S_{11}^{(1)}\left(6\psi_x-2\frac{d\phi_z}{dx}\right)\right\}-\delta\phi_x c_y^{(1)}\right]dx\\
&-\delta\phi_x(x_a,t)Q_3-\delta\phi_x(x_b,t)Q_{13}
\end{aligned} \tag{5.15c}$$

$$\begin{aligned}
0 = &\int_{x_a}^{x_b}\left[\delta\psi_x\left(m_3\frac{\partial^2 u}{\partial t^2}+m_4\frac{\partial^2\theta_x}{\partial t^2}+m_5\frac{\partial^2\phi_x}{\partial t^2}+m_6\frac{\partial^2\psi_x}{\partial t^2}\right)\right.\\
&+\frac{\partial\delta\psi_x}{\partial x}\left\{A_{11}^{(3)}\left(\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right)+A_{11}^{(4)}\frac{d\theta_x}{dx}+A_{11}^{(5)}\frac{d\phi_x}{dx}+A_{11}^{(6)}\frac{d\psi_x}{dx}\right.\\
&\left.+A_{12}^{(3)}\theta_z+A_{12}^{(4)}2\phi_z-X_T^{(3)}\right\}+3\delta\psi_x\left\{B_{11}^{(2)}\left(\theta_x+\frac{dw}{dx}\right)+B_{11}^{(3)}\left(2\phi_x+\frac{d\theta_z}{dx}\right)\right.\\
&\left.+B_{11}^{(4)}\left(3\psi_x+\frac{d\phi_z}{dx}\right)\right\}-\delta\psi_x F_x^{(3)}+\frac{3}{2}\frac{\partial\delta\psi_x}{\partial x}\left\{\frac{1}{4}S_{11}^{(2)}\left(\frac{d\theta_x}{dx}-\frac{d^2w}{dx^2}\right)\right.\\
&\left.+\frac{1}{4}S_{11}^{(3)}\left(2\frac{d\phi_x}{dx}-\frac{d^2\theta_z}{dx^2}\right)+\frac{1}{4}S_{11}^{(4)}\left(3\frac{d\psi_x}{dx}-\frac{d^2\phi_z}{dx^2}\right)\right\}\\
&\left.+3\delta\psi_x\left\{\frac{1}{4}S_{11}^{(1)}\left(2\phi_x-\frac{d\theta_z}{dx}\right)+\frac{1}{4}S_{11}^{(2)}\left(6\psi_x-2\frac{d\phi_z}{dx}\right)\right\}-\frac{3}{2}\delta\psi_x c_y^{(2)}\right]dx\\
&-\delta\psi_x(x_a,t)Q_4-\delta\psi_x(x_b,t)Q_{14}
\end{aligned} \tag{5.15d}$$

$$\begin{aligned}
0 = &\int_{x_a}^{x_b}\left[\delta w\left(m_0\frac{\partial^2 w}{\partial t^2}+m_1\frac{\partial^2\theta_z}{\partial t^2}+m_2\frac{\partial^2\phi_z}{\partial t^2}\right)+\frac{d\delta w}{dx}\frac{\partial w}{\partial x}\left\{A_{11}^{(0)}\left(\frac{du}{dx}+\frac{1}{2}\left(\frac{dw}{dx}\right)^2\right)\right.\right.\\
&\left.+A_{11}^{(1)}\frac{d\theta_x}{dx}+A_{11}^{(2)}\frac{d\phi_x}{dx}+A_{11}^{(3)}\frac{d\psi_x}{dx}+A_{12}^{(0)}\theta_z+A_{12}^{(1)}2\phi_z-X_T^{(0)}\right\}\\
&+\frac{\partial\delta w}{\partial x}\left\{B_{11}^{(0)}\left(\theta_x+\frac{dw}{dx}\right)+B_{11}^{(1)}\left(2\phi_x+\frac{d\theta_z}{dx}\right)+B_{11}^{(2)}\left(3\psi_x+\frac{d\phi_z}{dx}\right)\right\}\\
&-\delta w F_z-\frac{1}{2}\frac{\partial^2\delta w}{\partial x^2}\left\{\frac{1}{4}S_{11}^{(0)}\left(\frac{d\theta_x}{dx}-\frac{d^2w}{dx^2}\right)+\frac{1}{4}S_{11}^{(1)}\left(2\frac{d\phi_x}{dx}-\frac{d^2\theta_z}{dx^2}\right)\right.\\
&\left.\left.+\frac{1}{4}S_{11}^{(2)}\left(3\frac{d\psi_x}{dx}-\frac{d^2\phi_z}{dx^2}\right)\right\}-\delta w\frac{1}{2}\frac{\partial c_y^{(0)}}{\partial x}\right]dx
\end{aligned}$$

$$-\delta w(x_a,t)Q_5 - \frac{\partial \delta w}{\partial x}(x_a,t)Q_6 - \delta w(x_b,t)Q_{15} - \frac{\partial \delta w}{\partial x}(x_b,t)Q_{16} \tag{5.15e}$$

$$\begin{aligned}
0 \;=\; & \int_{x_a}^{x_b} \left[\delta\theta_z \left(m_1 \frac{\partial^2 w}{\partial t^2} + m_2 \frac{\partial^2 \theta_z}{\partial t^2} + m_3 \frac{\partial^2 \phi_z}{\partial t^2} \right) + \frac{\partial \delta\theta_z}{\partial x} \left\{ B_{11}^{(1)} \left(\theta_x + \frac{dw}{dx} \right) \right.\right. \\
& + B_{11}^{(2)} \left(2\phi_x + \frac{d\theta_z}{dx} \right) + B_{11}^{(3)} \left(3\psi_x + \frac{d\phi_z}{dx} \right) \Bigg\} + \delta\theta_z \left\{ A_{12}^{(0)} \left(\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right) \right. \\
& + A_{12}^{(1)} \frac{d\theta_x}{dx} + A_{12}^{(2)} \frac{d\phi_x}{dx} + A_{12}^{(3)} \frac{d\psi_x}{dx} + A_{11}^{(0)} \theta_z + A_{11}^{(1)} 2\phi_z - Z_T^{(0)} \Bigg\} - \delta\theta_z F_z^{(1)} \\
& - \frac{1}{2} \frac{\partial^2 \delta\theta_z}{\partial x^2} \left\{ \frac{1}{4} S_{11}^{(1)} \left(\frac{d\theta_x}{dx} - \frac{d^2 w}{dx^2} \right) + \frac{1}{4} S_{11}^{(2)} \left(2\frac{d\phi_x}{dx} - \frac{d^2\theta_z}{dx^2} \right) \right. \\
& + \frac{1}{4} S_{11}^{(3)} \left(3\frac{d\psi_x}{dx} - \frac{d^2\phi_z}{dx^2} \right) \Bigg\} - \frac{1}{2} \frac{\partial \delta\theta_z}{\partial x} \left\{ \frac{1}{4} S_{11}^{(0)} \left(2\phi_x - \frac{d\theta_z}{dx} \right) \right. \\
& + \frac{1}{4} S_{11}^{(1)} \left(6\psi_x - 2\frac{d\phi_z}{dx} \right) \Bigg\} - \frac{1}{2} \delta\theta_z \frac{\partial c_y^{(1)}}{\partial x} \Bigg] \, dx \\
& - \delta\theta_z(x_a,t)Q_7 - \frac{\partial \delta\theta_z}{\partial x}(x_a,t)Q_8 - \delta\theta_z(x_b,t)Q_{17} - \frac{\partial \delta\theta_z}{\partial x}(x_b,t)Q_{18}
\end{aligned} \tag{5.15f}$$

$$\begin{aligned}
0 \;=\; & \int_{x_a}^{x_b} \left[\delta\phi_z \left(m_2 \frac{\partial^2 w}{\partial t^2} + m_3 \frac{\partial^2 \theta_z}{\partial t^2} + m_4 \frac{\partial^2 \phi_z}{\partial t^2} \right) + \frac{\partial \delta\phi_z}{\partial x} \left\{ B_{11}^{(2)} \left(\theta_x + \frac{dw}{dx} \right) \right.\right. \\
& + B_{11}^{(3)} \left(2\phi_x + \frac{d\theta_z}{dx} \right) + B_{11}^{(4)} \left(3\psi_x + \frac{d\phi_z}{dx} \right) \Bigg\} + 2\delta\phi_z \left\{ A_{12}^{(1)} \left(\frac{du}{dx} + \frac{1}{2} \left(\frac{dw}{dx} \right)^2 \right) \right. \\
& + A_{12}^{(2)} \frac{d\theta_x}{dx} + A_{12}^{(3)} \frac{d\phi_x}{dx} + A_{12}^{(4)} \frac{d\psi_x}{dx} + A_{11}^{(1)} \theta_z + A_{11}^{(2)} 2\phi_z - Z_T^{(1)} \Bigg\} - \delta\phi_z F_z^{(2)} \\
& - \frac{1}{2} \frac{\partial^2 \delta\phi_z}{\partial x^2} \left\{ \frac{1}{4} S_{11}^{(2)} \left(\frac{d\theta_x}{dx} - \frac{d^2 w}{dx^2} \right) + \frac{1}{4} S_{11}^{(3)} \left(2\frac{d\phi_x}{dx} - \frac{d^2\theta_z}{dx^2} \right) \right. \\
& + \frac{1}{4} S_{11}^{(4)} \left(3\frac{d\psi_x}{dx} - \frac{d^2\phi_z}{dx^2} \right) \Bigg\} \\
& - \frac{\partial \delta\phi_z}{\partial x} \left\{ \frac{1}{4} S_{11}^{(1)} \left(2\phi_x - \frac{d\theta_z}{dx} \right) + \frac{1}{4} S_{11}^{(2)} \left(6\psi_x - 2\frac{d\phi_z}{dx} \right) \right\} - \frac{1}{2} \delta\phi_z \frac{\partial c_y^{(2)}}{\partial x} \Bigg] \, dx \\
& - \delta\phi_z(x_a,t)Q_9 - \frac{\partial \delta\phi_z}{\partial x}(x_a,t)Q_{10} - \delta\phi_z(x_b,t)Q_{19} - \frac{\partial \delta\phi_z}{\partial x}(x_b,t)Q_{20}
\end{aligned} \tag{5.15g}$$

6. FINITE ELEMENT MODELS

6.1. Euler-Bernoulli Beam Theory

6.1.1. Conventional beam

Axial displacement u and transverse displacement w are approximated using linear and Hermite cubic interpolations, respectively.

$$u(x) \approx \sum_{j=1}^{2} u_j(t)\psi_j(x), \quad w(x) \approx \sum_{j=1}^{4} \bar{\Delta}_j(t)\varphi_j(x) \tag{6.1}$$

where $\psi_j(x)$ are the linear polynomials, $\varphi_j(x)$ are the Hermite cubic polynomials, (u_1, u_2) are the nodal values of u at x_a and x_b, respectively, and $\bar{\Delta}_j$ are the nodal values associated with w (see Reddy [45])

$$\bar{\Delta}_1(t) = w(x_a, t), \quad \bar{\Delta}_3(t) = w(x_b, t), \quad \bar{\Delta}_2(t) = \theta_x(x_a, t), \quad \bar{\Delta}_4(t) = \theta_x(x_b, t) \tag{6.2}$$

where $\theta_x \equiv -dw/dx$. By substituting Eq.(6.1) for u and w and putting $\delta u = \psi_i$ and $\delta w = \varphi_i$ into the virtual work statements in Eqs. (5.3a) and (5.3b), the finite element equations can be obtained as

$$\begin{bmatrix} \mathbf{M}^{11} & \mathbf{M}^{12} \\ \mathbf{M}^{21} & \mathbf{M}^{22} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}} \\ \ddot{\bar{\boldsymbol{\Delta}}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} \\ \mathbf{K}^{21} & \mathbf{K}^{22} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \bar{\boldsymbol{\Delta}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^{1} \\ \mathbf{F}^{2} \end{Bmatrix} \tag{6.3}$$

where,

$$
\begin{aligned}
M_{ij}^{11} &= \int_{x_a}^{x_b} I_0 \psi_i \psi_j \, dx, \qquad\qquad M_{ij}^{12} = -\int_{x_a}^{x_b} I_1 \psi_i \frac{\partial \varphi_j}{\partial x} \, dx \\
M_{ij}^{21} &= -\int_{x_a}^{x_b} I_1 \frac{\partial \varphi_i}{\partial x} \psi_j \, dx, \qquad M_{ij}^{22} = \int_{x_a}^{x_b} \left(I_0 \varphi_i \varphi_j + I_2 \frac{\partial \varphi_i}{\partial x} \frac{\partial \varphi_j}{\partial x} \right) dx, \\
K_{ij}^{11} &= \int_{x_a}^{x_b} A_{xx} \frac{d\psi_i}{dx} \frac{d\psi_j}{dx} \, dx, \quad F_i^1 = \int_{x_a}^{x_b} \frac{d\psi_i}{dx} N_{xx}^T \, dx + \psi_i(x_a) Q_1 + \psi_i(x_b) Q_4 \\
K_{ij}^{12} &= -\int_{x_a}^{x_b} B_{xx} \frac{d\psi_i}{dx} \frac{d^2\varphi_j}{dx^2} \, dx + \frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\psi_i}{dx} \frac{d\varphi_j}{dx} \, dx \\
K_{ij}^{21} &= -\int_{x_a}^{x_b} B_{xx} \frac{d^2\varphi_i}{dx^2} \frac{d\psi_j}{dx} \, dx + \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\varphi_i}{dx} \frac{d\psi_j}{dx} \, dx \\
K_{ij}^{22} &= \int_{x_a}^{x_b} \left[\frac{A_{xx}}{2} \left(\frac{dw}{dx} \right)^2 \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} - B_{xx} \frac{dw}{dx} \left(\frac{1}{2} \frac{d^2\varphi_i}{dx^2} \frac{d\varphi_j}{dx} + \frac{d\varphi_i}{dx} \frac{d^2\varphi_j}{dx^2} \right) \right] dx \\
&\quad + \int_{x_a}^{x_b} D_{xx} \frac{d^2\varphi_i}{dx^2} \frac{d^2\varphi_j}{dx^2} \, dx \\
F_i^2 &= \int_{x_a}^{x_b} \left(-\frac{d^2\varphi_i}{dx^2} M_{xx}^T + \frac{d\varphi_i}{dx} \frac{dw}{dx} N_{xx}^T + \varphi_i q \right) dx \\
&\quad + \varphi_i(x_a) Q_2 + \left(-\frac{d\varphi_i}{dx} \right)_{x_a} Q_3 + \varphi_i(x_b) Q_5 + \left(-\frac{d\varphi_i}{dx} \right)_{x_b} Q_6
\end{aligned}
\tag{6.4}
$$

Clearly, the stiffness matrix is not symmetric, i.e., $K_{ij}^{\alpha\beta} \neq K_{ji}^{\beta\alpha}$ for the nonlinear case. Further, there are two sources of the coupling between the axial displacement u and the transverse displacement w: first, the coupling is due to the extensional-bending coefficient B, and it is independent of the von Kármán nonlinearity; second, the coupling is due to the von Kármán nonlinearity, which is independent of the coupling coefficient B. Of course, the coefficient B has a stronger coupling in the presence of the von Kármán nonlinearity. The equations of motion (6.3) can be expressed in the standard form

$$
\mathbf{M}\ddot{\mathbf{\Delta}} + \mathbf{K}\mathbf{\Delta} = \mathbf{F} \tag{6.5}
$$

Full discretization using the Newmark scheme reduces the finite element Eq. (6.5) to (see Reddy [44, 45])

$$
\hat{\mathbf{K}}_{s+1}(\mathbf{\Delta}_{s+1})\mathbf{\Delta}_{s+1} = \hat{\mathbf{F}}_{s,s+1} \tag{6.6}
$$

where

$$
\hat{\mathbf{K}}_{s+1}(\mathbf{\Delta}_{s+1}) = \mathbf{K}_{s+1}(\mathbf{\Delta}_{s+1}) + a_3 \mathbf{M}_{s+1}, \quad \hat{\mathbf{F}}_{s,s+1} = \mathbf{F}_{s+1} + \mathbf{M}_{s+1}\mathbf{A}_s \tag{6.7}
$$

$$\mathbf{A}_s = a_3\mathbf{\Delta}_s + a_4\dot{\mathbf{\Delta}}_s + a_5\ddot{\mathbf{\Delta}}_s$$

$$a_1 = \alpha\Delta t, \quad a_2 = (1-\alpha)\Delta t \tag{6.8}$$

$$a_3 = \frac{2}{\gamma(\Delta t)^2}, \quad a_4 = a_3\Delta t, \quad a_5 = \frac{1}{\gamma} - 1$$

α and γ being the parameters of the Newmark scheme. The nonlinear equations (6.3) are solved using Newton's iterative method (see Reddy [44]), which involves the computation of the coefficients of the element tangent stiffness matrix $\hat{\mathbf{T}}^e$. It is convenient to view $\hat{\mathbf{T}}^e$ as one that has structure similar to $\mathbf{K}^e$ in Eq. (6.3)

$$\begin{bmatrix} \hat{\mathbf{T}}^{11} & \hat{\mathbf{T}}^{12} \\ \hat{\mathbf{T}}^{21} & \hat{\mathbf{T}}^{22} \end{bmatrix}^{(r)} \left\{ \begin{array}{c} \delta\mathbf{\Delta}^1 \\ \delta\mathbf{\Delta}^2 \end{array} \right\}_{(s+1)}^{(r+1)} = - \left\{ \begin{array}{c} \mathbf{R}^1 \\ \mathbf{R}^2 \end{array} \right\}_{(s+1)}^{(r)} \tag{6.9}$$

where symbol $\delta\mathbf{\Delta}$ denotes the increment of the displacements from the rth iteration to the $(r+1)$st iteration. Also, note that $\mathbf{\Delta}^1 = \mathbf{u}$ and $\mathbf{\Delta}^2 = \bar{\mathbf{\Delta}}$. Then we can compute the components $\hat{\mathbf{T}}^{\alpha\beta}$ from the definition (evaluated at the rth iteration)

$$\hat{T}_{ij}^{\alpha\beta} = \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta}, \quad \alpha, \beta = 1, 2 \tag{6.10}$$

The components R_i^α of the residual vector $\mathbf{R}$ can be expressed as:

$$\begin{aligned} R_i^\alpha &= \sum_{\gamma=1}^{2}\sum_{p=1}^{N_\gamma} \hat{K}_{ip}^{\alpha\gamma}\Delta_p^\gamma - \hat{F}_i^\alpha = \sum_{p=1}^{2} \hat{K}_{ip}^{\alpha 1}\Delta_p^1 + \sum_{P=1}^{4} \hat{K}_{iP}^{\alpha 2}\Delta_P^2 - \hat{F}_i^\alpha \\ &= \sum_{p=1}^{2} \hat{K}_{ip}^{\alpha 1} u_p + \sum_{P=1}^{4} \hat{K}_{iP}^{\alpha 2}\bar{\Delta}_P - \hat{F}_i^\alpha \end{aligned} \tag{6.11}$$

where N_γ $(\gamma = 1, 2)$ denotes the number of element degrees of freedom [$N_1 = 2$ and $N_2 = 4$]. We have

$$\begin{aligned}
\hat{T}_{ij}^{\alpha\beta} &= \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta} = \frac{\partial}{\partial \Delta_j^\beta}\left(\sum_{\gamma=1}^{2}\sum_{p=1}^{n(\gamma)} \hat{K}_{ip}^{\alpha\gamma}\Delta_p^\gamma - \hat{F}_i^\alpha\right) \\
&= \sum_{\gamma=1}^{2}\sum_{p=1}^{n(\gamma)}\left(\hat{K}_{ip}^{\alpha\gamma}\frac{\partial \Delta_p^\gamma}{\partial \Delta_j^\beta} + \frac{\partial \hat{K}_{ip}^{\alpha\gamma}}{\partial \Delta_j^\beta}\Delta_p^\gamma\right) - \frac{\partial \hat{F}_i^\alpha}{\partial \Delta_j^\beta} \\
&= \hat{K}_{ij}^{\alpha\beta} + \sum_{p=1}^{2}\frac{\partial}{\partial \Delta_j^\beta}\left(\hat{K}_{ip}^{\alpha 1}\right)u_p + \sum_{P=1}^{4}\frac{\partial}{\partial \Delta_j^\beta}\left(\hat{K}_{iP}^{\alpha 2}\right)\bar{\Delta}_P - \frac{\partial \hat{F}_i^\alpha}{\partial \Delta_j^\beta} \qquad (6.12)
\end{aligned}$$

Since $\hat{F}_i^\alpha$ and $\hat{K}_{ij}^{\alpha\beta}$ depend at most only on w and not on u, we have

$$\hat{T}_{ij}^{\alpha 1} = \hat{K}_{ij}^{\alpha 1} \quad \text{for } \alpha = 1, 2 \qquad (6.13)$$

The tangent stiffness coefficients that are different from their counterparts are computed, noting that $\mathbf{M}$ and $\hat{F}_i^1$ are not functions of the current solution, as follows:

$$\begin{aligned}
\hat{T}_{ij}^{12} &= \hat{K}_{ij}^{12} + \sum_{P=1}^{4}\frac{\partial}{\partial \bar{\Delta}_j}\left(\hat{K}_{iP}^{12}\right)\bar{\Delta}_p \\
&= \hat{K}_{ij}^{12} + \int_{x_a}^{x_b}\frac{1}{2}A_{xx}\frac{d\psi_i}{dx}\frac{d\varphi_j}{dx}\left(\sum_{p=1}^{4}\frac{d\varphi_p}{dx}\bar{\Delta}_p\right)\,dx \\
&= \hat{K}_{ij}^{12} + \int_{x_a}^{x_b}\left(\frac{1}{2}A_{xx}\frac{\partial w}{\partial x}\right)\frac{d\psi_i}{dx}\frac{d\varphi_j}{dx}\,dx = \hat{K}_{ji}^{21} \qquad (6.14) \\
\hat{T}_{ij}^{22} &= \hat{K}_{ij}^{22} + \sum_{p=1}^{2}\frac{\partial}{\partial \bar{\Delta}_j}\left(\hat{K}_{ip}^{21}\right)u_p + \sum_{P=1}^{4}\frac{\partial}{\partial \bar{\Delta}_j}\left(\hat{K}_{ip}^{22}\right)\bar{\Delta}_p - \frac{\partial \hat{F}_i^2}{\partial \bar{\Delta}_j} \\
&= \hat{K}_{ij}^{22} + \sum_{p=1}^{2}\left[\int_{x_a}^{x_b} A_{xx}\frac{\partial}{\partial \bar{\Delta}_j}\left(\frac{\partial w}{\partial x}\right)\frac{d\varphi_i}{dx}\frac{d\psi_p}{dx}\,dx\right]u_p \\
&\quad + \sum_{p=1}^{4}\left[\int_{x_a}^{x_b}\frac{1}{2}A_{xx}\frac{\partial}{\partial \bar{\Delta}_j}\left(\frac{\partial w}{\partial x}\right)^2\frac{d\varphi_i}{dx}\frac{d\varphi_p}{dx}\,dx\right]\bar{\Delta}_p \\
&\quad - \sum_{p=1}^{4}\left[\int_{x_a}^{x_b} B_{xx}\frac{\partial}{\partial \bar{\Delta}_j}\left(\frac{\partial w}{\partial x}\right)\left(\frac{1}{2}\frac{d^2\varphi_i}{dx^2}\frac{d\varphi_p}{dx} + \frac{d\varphi_i}{dx}\frac{d^2\varphi_p}{dx^2}\right)\,dx\right]\bar{\Delta}_p \\
&\quad - \int_{x_a}^{x_b} N_{xx}^T\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \hat{K}_{ij}^{22} + \int_{x_a}^{x_b} A_{xx}\frac{\partial u}{\partial x}\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx + \int_{x_a}^{x_b} A_{xx}\left(\frac{\partial w}{\partial x}\right)^2\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx \\
&\quad - \int_{x_a}^{x_b} B_{xx}\left(\frac{1}{2}\frac{\partial w}{\partial x}\frac{d^2\varphi_i}{dx^2}\frac{d\varphi_j}{dx} + \frac{\partial^2 w}{\partial x^2}\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\right)dx \\
&\quad - \int_{x_a}^{x_b} N_{xx}^T\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx}\,dx \qquad (6.15)
\end{aligned}
$$

Clearly, the tangent stiffness matrix of functionally graded Euler–Bernoulli beam element with the von Kármán nonlinearity is symmetric.

6.1.2. Microstructure dependent beam

In case of microstructure dependent beam, if the axial displacement u and transverse displacement w are approximated as linear and Hermite cubic interpolation function as given in Eqs. (6.1) and (6.2), then the virtual work statements in Eqs. (5.6a) and (5.6b) can be given in the form of Eq. (6.3) and the mass matrix, stiffness matrix and the force vector can be given as

$$
\begin{aligned}
M_{ij}^{11} &= \int_{x_a}^{x_b} I_0\psi_i\psi_j\,dx, \qquad M_{ij}^{12} = -\int_{x_a}^{x_b} I_1\psi_i\frac{\partial\varphi_j}{\partial x}\,dx \\
M_{ij}^{21} &= -\int_{x_a}^{x_b} I_1\frac{\partial\varphi_i}{\partial x}\psi_j\,dx, \qquad M_{ij}^{22} = \int_{x_a}^{x_b}\left(I_0\varphi_i\varphi_j + I_2\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x}\right)dx, \\
K_{ij}^{11} &= \int_{x_a}^{x_b} A_{xx}\frac{d\psi_i}{dx}\frac{d\psi_j}{dx}\,dx, \quad F_i^1 = \int_{x_a}^{x_b}\frac{d\psi_i}{dx}N_{xx}^T\,dx + \psi_i(x_a)Q_1 + \psi_i(x_b)Q_4 \\
K_{ij}^{12} &= -\int_{x_a}^{x_b} B_{xx}\frac{d\psi_i}{dx}\frac{d^2\varphi_j}{dx^2}\,dx + \frac{1}{2}\int_{x_a}^{x_b} A_{xx}\frac{dw}{dx}\frac{d\psi_i}{dx}\frac{d\varphi_j}{dx}\,dx \\
K_{ij}^{21} &= -\int_{x_a}^{x_b} B_{xx}\frac{d^2\varphi_i}{dx^2}\frac{d\psi_j}{dx}\,dx + \int_{x_a}^{x_b} A_{xx}\frac{dw}{dx}\frac{d\varphi_i}{dx}\frac{d\psi_j}{dx}\,dx \\
K_{ij}^{22} &= \int_{x_a}^{x_b}\left[\frac{A_{xx}}{2}\left(\frac{dw}{dx}\right)^2\frac{d\varphi_i}{dx}\frac{d\varphi_j}{dx} - B_{xx}\frac{dw}{dx}\left(\frac{1}{2}\frac{d^2\varphi_i}{dx^2}\frac{d\varphi_j}{dx} + \frac{d\varphi_i}{dx}\frac{d^2\varphi_j}{dx^2}\right)\right]dx \\
&\quad + \int_{x_a}^{x_b}(D_{xx} + S_{xy})\frac{d^2\varphi_i}{dx^2}\frac{d^2\varphi_j}{dx^2}\,dx \\
F_i^2 &= \int_{x_a}^{x_b}\left(-\frac{d^2\varphi_i}{dx^2}M_{xx}^T + \frac{d\varphi_i}{dx}\frac{dw}{dx}N_{xx}^T + \varphi_i q\right)dx \\
&\quad + \varphi_i(x_a)Q_2 + \left(-\frac{d\varphi_i}{dx}\right)_{x_a}Q_3 + \varphi_i(x_b)Q_5 + \left(-\frac{d\varphi_i}{dx}\right)_{x_b}Q_6 \qquad (6.16)
\end{aligned}
$$

The stiffness matrix, as in the conventional FGM beam, is not symmetric. Again applying time approximation using Newmark scheme, Eq. of the form (6.6) can be obtained, which can further be written in the form of Eq. (6.9), in which the tangent matrix can be given as:

$$
\begin{aligned}
\hat{T}_{ij}^{\alpha 1} &= \hat{K}_{ij}^{\alpha 1} \quad \text{for } \alpha = 1, 2 \\
\hat{T}_{ij}^{12} &= \hat{K}_{ij}^{12} + \int_{x_a}^{x_b} \left(\frac{1}{2} A_{xx} \frac{\partial w}{\partial x} \right) \frac{d\psi_i}{dx} \frac{d\varphi_j}{dx}\, dx = \hat{K}_{ji}^{21} \\
\hat{T}_{ij}^{22} &= \hat{K}_{ij}^{22} + \int_{x_a}^{x_b} A_{xx} \frac{\partial u}{\partial x} \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx}\, dx + \int_{x_a}^{x_b} A_{xx} \left(\frac{\partial w}{\partial x} \right)^2 \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx}\, dx \\
&\quad - \int_{x_a}^{x_b} B_{xx} \left(\frac{1}{2} \frac{\partial w}{\partial x} \frac{d^2\varphi_i}{dx^2} \frac{d\varphi_j}{dx} + \frac{\partial^2 w}{\partial x^2} \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} \right) dx \\
&\quad - \int_{x_a}^{x_b} N_{xx}^T \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx}\, dx
\end{aligned} \tag{6.17}
$$

From the above expression, it is clear that the tangent matrix is symmetric as in the case of conventional Euler–Bernoulli FGM beam.

6.2. Timoshenko Beam Theory

6.2.1. Conventional beam

We assume Lagrange approximation of the all field variables (u, w, ϕ_x) independently

$$
u(x) \approx \sum_{j=1}^{m} u_j(t) \psi_j^{(1)}(x), \quad w(x) \approx \sum_{j=1}^{n} w_j(t) \psi_j^{(2)}(x), \quad \phi_x \approx \sum_{j=1}^{p} S_j(t) \psi_j^{(3)}(x) \tag{6.18}
$$

where $\psi_j^{(\alpha)}(x)$ are the Lagrange polynomials of different order used for the three variables. Substitution of Eq. (6.18) for (u, w, ϕ_x) and $\delta u = \psi_i^{(1)}$, $\delta w = \psi_i^{(2)}$, and $\delta \phi_x = \psi_i^{(3)}$ into the virtual work statements in Eqs. (5.9a)–(5.9c), we obtain the finite element equations as:

$$
\begin{bmatrix} \mathbf{M}^{11} & \mathbf{0} & \mathbf{M}^{13} \\ \mathbf{0} & \mathbf{M}^{22} & \mathbf{0} \\ \mathbf{M}^{31} & \mathbf{0} & \mathbf{M}^{33} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}} \\ \ddot{\bar{\mathbf{w}}} \\ \ddot{\mathbf{s}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \mathbf{K}^{13} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \mathbf{K}^{23} \\ \mathbf{K}^{31} & \mathbf{K}^{32} & \mathbf{K}^{33} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \bar{\mathbf{w}} \\ \mathbf{s} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ \mathbf{F}^3 \end{Bmatrix} \tag{6.19}
$$

where

$$
\begin{aligned}
M_{ij}^{11} &= \int_{x_a}^{x_b} I_0 \psi_i^{(1)} \psi_j^{(1)}\, dx, \quad M_{ij}^{22} = \int_{x_a}^{x_b} I_0 \psi_i^{(2)} \psi_j^{(2)}\, dx \quad M_{ij}^{33} = \int_{x_a}^{x_b} I_2 \psi_i^{(3)} \psi_j^{(3)}\, dx \\
M_{ij}^{13} &= \int_{x_a}^{x_b} I_1 \psi_i^{(1)} \psi_j^{(3)}\, dx, \quad M_{ij}^{31} = \int_{x_a}^{x_b} I_1 \psi_i^{(3)} \psi_j^{(1)}\, dx, \\
K_{ij}^{11} &= \int_{x_a}^{x_b} A_{xx} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(1)}}{dx}\, dx, \quad F_i^1 = \int_{x_a}^{x_b} \frac{d\psi_i^{(1)}}{dx} N_{xx}^T\, dx + \psi_i^{(1)}(x_a) Q_1 + \psi_i^{(1)}(x_b) Q_4 \\
K_{ij}^{12} &= \frac{1}{2} \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(2)}}{dx}\, dx, \quad K_{ij}^{13} = \int_{x_a}^{x_b} B_{xx} \frac{d\psi_i^{(1)}}{dx} \frac{d\psi_j^{(3)}}{dx}\, dx \\
K_{ij}^{21} &= \int_{x_a}^{x_b} A_{xx} \frac{dw}{dx} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(1)}}{dx}\, dx \\
K_{ij}^{22} &= \int_{x_a}^{x_b} \left[K_s S_{xz} \frac{d^2\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} + \frac{A_{xx}}{2} \left(\frac{dw}{dx} \right)^2 \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(2)}}{dx} \right] dx \\
K_{ij}^{23} &= \int_{x_a}^{x_b} \left(K_s S_{xz} \frac{d\psi^{(2)}}{dx} \psi_j^{(3)} + B_{xx} \frac{dw}{dx} \frac{d\psi_i^{(2)}}{dx} \frac{d\psi_j^{(3)}}{dx} \right) dx \\
F_i^2 &= \int_{x_a}^{x_b} \frac{d\psi_i^{(2)}}{dx} \frac{dw}{dx} N_{xx}^T\, dx + \psi_i^{(2)}(x_a) Q_2 + \psi_i^{(2)}(x_b) Q_5 \\
K_{ij}^{31} &= \int_{x_a}^{x_b} B_{xx} \frac{d\psi_i^{(3)}}{dx} \frac{d\psi_j^{(1)}}{dx}\, dx \\
K_{ij}^{32} &= \int_{x_a}^{x_b} \left(K_s S_{xz} \psi_i^{(3)} \frac{d\psi_j^{(2)}}{dx} + \frac{B_{xx}}{2} \frac{dw}{dx} \frac{d\psi_i^{(3)}}{dx} \frac{d\psi_j^{(2)}}{dx} \right) dx \\
K_{ij}^{33} &= \int_{x_a}^{x_b} \left(K_s S_{xz} \psi_i^{(3)} \psi_j^{(3)} + D_{xx} \frac{d\psi_i^{(3)}}{dx} \frac{d\psi_j^{(3)}}{dx} \right) dx \\
F_i^3 &= \int_{x_a}^{x_b} \frac{d\psi_i^{(3)}}{dx} M_{xx}^T\, dx + \psi_i^{(3)}(x_a) Q_3 + \psi_i^{(3)}(x_b) Q_6
\end{aligned}
\tag{6.20}
$$

Once again, Eq. (6.19) can be written in the standard form as Eq. (6.5) and by applying time approximation, nonlinear algebraic equations of the form of Eq. (6.6) can be obtained. For the Newton's iterative procedure, the tangent must be computed stiffness matrix $\hat{\mathbf{T}}^e$, which has a structure similar to $\mathbf{K}^e$ in Eq. (6.19).

$$\begin{bmatrix} \hat{\mathbf{T}}^{11} & \hat{\mathbf{T}}^{12} & \hat{\mathbf{T}}^{13} \\ \hat{\mathbf{T}}^{21} & \hat{\mathbf{T}}^{22} & \hat{\mathbf{T}}^{23} \\ \hat{\mathbf{T}}^{31} & \hat{\mathbf{T}}^{32} & \hat{\mathbf{T}}^{33} \end{bmatrix} \begin{Bmatrix} \delta\mathbf{u} \\ \delta\mathbf{w} \\ \delta\mathbf{s} \end{Bmatrix}^{(r+1)}_{(s+1)} = -\begin{Bmatrix} \mathbf{R}^1 \\ \mathbf{R}^2 \\ \mathbf{R}^3 \end{Bmatrix}^{(r)} \tag{6.21}$$

where

$$\hat{T}_{ij}^{\alpha\beta} = \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta}, \quad \alpha,\beta = 1,2,3; \quad \mathbf{\Delta}^1 = \mathbf{u}, \quad \mathbf{\Delta}^2 = \mathbf{w}, \quad \mathbf{\Delta}^3 = \mathbf{s} \tag{6.22}$$

and

$$\begin{aligned} \hat{R}_i^\alpha &= \sum_{\gamma=1}^{3}\sum_{k=1}^{N_\gamma} \hat{K}_{ik}^{\alpha\gamma}\Delta_k^\gamma - \hat{F}_i^\alpha \\ &= \sum_{k=1}^{m} \hat{K}_{ik}^{\alpha 1} u_k + \sum_{k=1}^{n} \hat{K}_{ik}^{\alpha 2} w_k + \sum_{k=1}^{p} \hat{K}_{ik}^{\alpha 3} s_k - \hat{F}_i^\alpha \end{aligned} \tag{6.23}$$

where $N_1 = m$, $N_2 = n$, and $N_3 = p$. Since $\hat{F}_i^\alpha$ and $\hat{K}_{ij}^{\alpha\beta}$ depend, at the most, only on w and not on u and ϕ_x, we have

$$\hat{T}_{ij}^{\alpha 1} = \hat{K}_{ij}^{\alpha 1}, \quad \hat{T}_{ij}^{\alpha 3} = \hat{K}_{ij}^{\alpha 3} \quad \text{for } \alpha = 1,2,3 \tag{6.24}$$

Thus, only tangent stiffness coefficients that need to be computed are $\hat{T}_{ij}^{12}$, $\hat{T}_{ij}^{22}$, and $\hat{T}_{ij}^{32}$. We have

$$\begin{aligned} \hat{T}_{ij}^{12} &= \hat{K}_{ij}^{12} + \sum_{k=1}^{n} \frac{\partial}{\partial w_j}\left(\hat{K}_{ik}^{12}\right) w_k \\ &= \hat{K}_{ij}^{12} + \int_{x_a}^{x_b} \frac{1}{2} A_{xx} \frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(2)}}{dx}\left(\sum_{k=1}^{n}\frac{d\psi_k^{(2)}}{dx} w_k\right) dx \\ &= \hat{K}_{ij}^{12} + \frac{1}{2}\int_{x_a}^{x_b} A_{xx}\frac{dw}{dx}\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(2)}}{dx}\, dx = \hat{K}_{ji}^{21} \end{aligned} \tag{6.25}$$

$$\begin{aligned} \hat{T}_{ij}^{22} &= \hat{K}_{ij}^{22} + \sum_{k=1}^{m} \frac{\partial}{\partial w_j}\left(\hat{K}_{ik}^{21}\right) u_k + \sum_{k=1}^{n} \frac{\partial}{\partial w_j}\left(\hat{K}_{ik}^{22}\right) w_k + \sum_{k=1}^{p} \frac{\partial}{\partial w_j}\left(\hat{K}_{ik}^{23}\right) s_k - \frac{\partial \hat{F}_i^2}{\partial w_j} \\ &= \hat{K}_{ij}^{22} + \int_{x_a}^{x_b}\left[A_{xx}\frac{du}{dx} + A_{xx}\left(\frac{dw}{dx}\right)^2 + B_{xx}\frac{d\phi_x}{dx} - N_{xx}^T\right]\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx}\, dx \\ \hat{T}_{ij}^{32} &= \hat{K}_{ij}^{32} + \frac{1}{2}\int_{x_a}^{x_b} B_{xx}\frac{dw}{dx}\frac{d\psi_i^{(3)}}{dx}\frac{d\psi_j^{(2)}}{dx} dx = \hat{K}_{ji}^{23} \end{aligned} \tag{6.26}$$

Once again, it is noted that the tangent stiffness matrix of functionally graded Timoshenko beam element with the von Kármán nonlinearity is symmetric.

6.2.2. Microstructure dependent beam

we approximate the field variables (u, ϕ_x) by Lagrange interpolation function and w by Hermite interpolation function as:

$$u(x) \approx \sum_{j=1}^{m} u_j(t)\psi_j^{(1)}(x), \quad w(x) \approx \sum_{j=1}^{n} w_j(t)\psi_j^{(2)}(x), \quad \phi_x \approx \sum_{j=1}^{p} S_j(t)\psi_j^{(3)}(x) \qquad (6.27)$$

where $\psi_j^{(1)}(x)$ and $\psi_j^{(3)}(x)$ are the Lagrange polynomials of different order used for u and ϕ_x respectively and $\psi_j^{(1)}(x)$ is Hermite interpolation function used for w as derivative of w $(\partial w/\partial x)$ and w both are primary variable in microstructure dependent Timoshenko beam. Substitution of Eq. (6.27) for (u, w, ϕ_x) and $\delta u = \psi_i^{(1)}$, $\delta w = \psi_i^{(2)}$, and $\delta\phi_x = \psi_i^{(3)}$ into the virtual work statements in Eqs. (5.12a)–(5.12c), the finite element equations can be obtained in the form of Eq. (6.19). where the components of stiffness matrix, mass matrix and force vector can be given as:

$$\begin{aligned}
K_{ij}^{11} &= \int_{x_a}^{x_b} A_{xx}\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(1)}}{dx}\,dx, \qquad K_{ij}^{12} = \frac{1}{2}\int_{x_a}^{x_b} A_{xx}\frac{dw}{dx}\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(2)}}{dx}\,dx \\
K_{ij}^{13} &= \int_{x_a}^{x_b} B_{xx}\frac{d\psi_i^{(1)}}{dx}\frac{d\psi_j^{(3)}}{dx}\,dx, \quad K_{ij}^{21} = \int_{x_a}^{x_b} A_{xx}\frac{dw}{dx}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(1)}}{dx}\,dx \\
K_{ij}^{22} &= \int_{x_a}^{x_b}\left[K_sS_{xz}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx} + \frac{A_{xx}}{2}\left(\frac{dw}{dx}\right)^2\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(2)}}{dx} + \frac{1}{4}S_{xy}\frac{d^2\psi_i^{(2)}}{dx^2}\frac{d^2\psi_j^{(2)}}{dx^2}\right]dx \\
K_{ij}^{23} &= \int_{x_a}^{x_b}\left[K_sS_{xz}\frac{d\psi^{(2)}}{dx}\psi_j^{(3)} + B_{xx}\frac{dw}{dx}\frac{d\psi_i^{(2)}}{dx}\frac{d\psi_j^{(3)}}{dx} - \frac{1}{4}S_{xy}\frac{d^2\psi_i^{(2)}}{dx^2}\frac{d\psi_j^{(3)}}{dx}\right]dx \qquad (6.28) \\
K_{ij}^{31} &= \int_{x_a}^{x_b} B_{xx}\frac{d\psi_i^{(3)}}{dx}\frac{d\psi_j^{(1)}}{dx}\,dx \\
K_{ij}^{32} &= \int_{x_a}^{x_b}\left[K_sS_{xz}\psi_i^{(3)}\frac{d\psi_j^{(2)}}{dx} + \frac{B_{xx}}{2}\frac{dw}{dx}\frac{d\psi_i^{(3)}}{dx}\frac{d\psi_j^{(2)}}{dx} - \frac{1}{4}S_{xy}\frac{d\psi_i^{(3)}}{dx}\frac{d^2\psi_j^{(2)}}{dx^2}\right]dx \\
K_{ij}^{33} &= \int_{x_a}^{x_b}\left[K_sS_{xz}\psi_i^{(3)}\psi_j^{(3)} + D_{xx}\frac{d\psi_i^{(3)}}{dx}\frac{d\psi_j^{(3)}}{dx} + \frac{1}{4}S_{xy}\frac{d\psi_i^{(3)}}{dx}\frac{d\psi_j^{(3)}}{dx}\right]dx
\end{aligned}$$

$$
\begin{aligned}
M_{ij}^{11} &= \int_{x_a}^{x_b} I_0 \psi_i^{(1)} \psi_j^{(1)}\, dx, \quad M_{ij}^{13} = \int_{x_a}^{x_b} I_1 \psi_i^{(1)} \psi_j^{(3)}\, dx, \\
M_{ij}^{22} &= \int_{x_a}^{x_b} I_0 \psi_i^{(2)} \psi_j^{(2)}\, dx, \quad M_{ij}^{31} = \int_{x_a}^{x_b} I_1 \psi_i^{(3)} \psi_j^{(1)}\, dx, \\
M_{ij}^{33} &= \int_{x_a}^{x_b} I_2 \psi_i^{(3)} \psi_j^{(3)}\, dx
\end{aligned} \tag{6.29}
$$

$$
\begin{aligned}
F_i^1 &= \int_{x_a}^{x_b} \frac{d\psi_i^{(1)}}{dx} N_{xx}^T\, dx + \psi_i^{(1)}(x_a) Q_1 + \psi_i^{(1)}(x_b) Q_4 \\
F_i^2 &= \int_{x_a}^{x_b} \frac{d\psi_i^{(2)}}{dx} \frac{dw}{dx} N_{xx}^T\, dx + \psi_i^{(2)}(x_a) Q_2 + \psi_i^{(2)}(x_b) Q_5 \\
F_i^3 &= \int_{x_a}^{x_b} \frac{d\psi_i^{(3)}}{dx} M_{xx}^T\, dx + \psi_i^{(3)}(x_a) Q_3 + \psi_i^{(3)}(x_b) Q_6
\end{aligned} \tag{6.30}
$$

Further, by applying time approximation, the finite element equation can be written as the nonlinear algebraic equations of the form (6.6). Again applying the Newton's iterative procedure, nonlinear equation can be written in the form of Eq. (6.21) and the tangent matrix, $\hat{\mathbf{T}}^e$ can be given as

$$
\hat{T}_{ij}^{\alpha\beta} = \frac{\partial R_i^\alpha}{\partial \Delta_j^\beta}, \quad \alpha, \beta = 1, 2, 3; \quad \boldsymbol{\Delta}^1 = \mathbf{u}, \quad \boldsymbol{\Delta}^2 = \mathbf{w}, \quad \boldsymbol{\Delta}^3 = \mathbf{s} \tag{6.31}
$$

where

$$
\begin{aligned}
\hat{R}_i^\alpha &= \sum_{\gamma=1}^{3} \sum_{k=1}^{N_\gamma} \hat{K}_{ik}^{\alpha\gamma} \Delta_k^\gamma - \hat{F}_i^\alpha \\
&= \sum_{k=1}^{m} \hat{K}_{ik}^{\alpha 1} u_k + \sum_{k=1}^{n} \hat{K}_{ik}^{\alpha 2} w_k + \sum_{k=1}^{p} \hat{K}_{ik}^{\alpha 3} s_k - \hat{F}_i^\alpha
\end{aligned} \tag{6.32}
$$

where $N_1 = m$, $N_2 = n$, and $N_3 = p$. Then we have

$$
\begin{aligned}
\hat{T}^{\alpha 1}_{ij} &= \hat{K}^{\alpha 1}_{ij}, \quad \hat{T}^{\alpha 3}_{ij} = \hat{K}^{\alpha 3}_{ij} \quad \text{for } \alpha = 1,2,3 \\
\hat{T}^{12}_{ij} &= \hat{K}^{12}_{ij} + \sum_{k=1}^{n} \frac{\partial}{\partial w_j}\left(\hat{K}^{12}_{ik}\right) w_k \\
&= \hat{K}^{12}_{ij} + \int_{x_a}^{x_b} \frac{1}{2} A_{xx} \frac{d\psi^{(1)}_i}{dx}\frac{d\psi^{(2)}_j}{dx}\left(\sum_{k=1}^{n}\frac{d\psi^{(2)}_k}{dx} w_k\right) dx \\
&= \hat{K}^{12}_{ij} + \frac{1}{2}\int_{x_a}^{x_b} A_{xx}\frac{dw}{dx}\frac{d\psi^{(1)}_i}{dx}\frac{d\psi^{(2)}_j}{dx}\, dx = \hat{K}^{21}_{ji} \qquad (6.33) \\
\hat{T}^{22}_{ij} &= \hat{K}^{22}_{ij} + \sum_{k=1}^{m}\frac{\partial}{\partial w_j}\left(\hat{K}^{21}_{ik}\right) u_k + \sum_{k=1}^{n}\frac{\partial}{\partial w_j}\left(\hat{K}^{22}_{ik}\right) w_k + \sum_{k=1}^{p}\frac{\partial}{\partial w_j}\left(\hat{K}^{23}_{ik}\right) s_k - \frac{\partial \hat{F}^2_i}{\partial w_j} \\
&= \hat{K}^{22}_{ij} + \int_{x_a}^{x_b}\left[A_{xx}\frac{du}{dx} + A_{xx}\left(\frac{dw}{dx}\right)^2 + B_{xx}\frac{d\phi_x}{dx} - N^T_{xx}\right]\frac{d\psi^{(2)}_i}{dx}\frac{d\psi^{(2)}_j}{dx}\, dx \\
\hat{T}^{32}_{ij} &= \hat{K}^{32}_{ij} + \frac{1}{2}\int_{x_a}^{x_b} B_{xx}\frac{dw}{dx}\frac{d\psi^{(3)}_i}{dx}\frac{d\psi^{(2)}_j}{dx} dx = \hat{K}^{23}_{ji}
\end{aligned}
$$

Once again, it is noted that the tangent stiffness matrix of functionally graded microstructure dependent Timoshenko beam element with the von Kármán nonlinearity is also symmetric.

6.3. General Third-Order Beam Theory

For finite element equation of general third order beam theory, we approximate the field variables $(u, \theta_x, \phi_x, \psi_x)$ by Lagrange interpolation function of different order and (w, θ_z, ϕ_z) by Hermite polynomial as:

$$
\begin{aligned}
&u(x) \approx \sum_{j=1}^{n_1} u_j(t)\varphi^{(1)}(x), \quad \theta_x(x) \approx \sum_{j=1}^{n_2} S^{(1)}_j(t)\varphi^{(2)}(x), \quad \phi_x(x) \approx \sum_{j=1}^{n_3} S^{(2)}_j(t)\varphi^{(3)}(x), \\
&\psi_x(x) \approx \sum_{j=1}^{n_4} S^{(3)}_j(t)\varphi^{(4)}(x), \quad w(x) \approx \sum_{j=1}^{n_5} w_j(t)\varphi^{(5)}(x), \quad \theta_z(x) \approx \sum_{j=1}^{n_6} S^{(4)}_j(t)\varphi^{(6)}(x), \\
&\phi_z(x) \approx \sum_{j=1}^{n_7} S^{(5)}_j(t)\varphi^{(7)}(x) \qquad (6.34)
\end{aligned}
$$

where $\varphi^{(i)}(x)$ for $i = 1,2,3,4$ are Lagrange polynomials of different order used for $u, \theta_x, \phi_x$and ψ_x respectively and $\varphi^{(i)}(x)$ for $i = 5,6,7$ are Hermite polynomials used for w, θ_z and ϕ_z respectively. $u_j(t), S_j^{(1)}(t), S_j^{(2)}(t), S_j^{(3)}(t), w_j(t), S_j^{(4)}(t), S_j^{(5)}(t)$ are the nodal values of $u, \theta_x, \phi_x, \psi_x, w, \theta_z, \phi_z$ respectively. Substituting $u, \theta_x, \phi_x, \psi_x, w, \theta_z, \phi_z$ from (6.34) and $\delta u = \varphi_i^{(1)}, \delta\theta_x = \varphi_i^{(2)}, \delta\phi_x = \varphi_i^{(3)}, \delta\psi_x = \varphi_i^{(4)}, \delta w = \varphi_i^{(5)}, \delta\theta_z = \varphi_i^{(6)}$ and $\delta\phi_z = \varphi_i^{(7)}$ into the weak statements (5.15a)–(5.15g), we obtain the finite element equation as

$$\begin{bmatrix} \mathbf{M}^{11} & \mathbf{M}^{12} & \dots & \mathbf{M}^{17} \\ \mathbf{M}^{21} & \mathbf{M}^{22} & \dots & \mathbf{M}^{27} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{M}^{71} & \mathbf{M}^{72} & \dots & \mathbf{M}^{77} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{\Delta}}^1 \\ \ddot{\mathbf{\Delta}}^2 \\ \vdots \\ \ddot{\mathbf{\Delta}}^7 \end{Bmatrix} + \begin{bmatrix} \mathbf{K}^{11} & \mathbf{K}^{12} & \dots & \mathbf{K}^{17} \\ \mathbf{K}^{21} & \mathbf{K}^{22} & \dots & \mathbf{K}^{27} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}^{71} & \mathbf{K}^{72} & \dots & \mathbf{K}^{77} \end{bmatrix} \begin{Bmatrix} \mathbf{\Delta}^1 \\ \mathbf{\Delta}^2 \\ \vdots \\ \mathbf{\Delta}^7 \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}^1 \\ \mathbf{F}^2 \\ \vdots \\ \mathbf{F}^7 \end{Bmatrix} \tag{6.35}$$

where $\mathbf{\Delta}^1 = \mathbf{u}$, $\mathbf{\Delta}^2 = \mathbf{s}^{(1)}$, $\mathbf{\Delta}^3 = \mathbf{s}^{(2)}$, $\mathbf{\Delta}^4 = \mathbf{s}^{(3)}$, $\mathbf{\Delta}^5 = \mathbf{w}$, $\mathbf{\Delta}^6 = \mathbf{s}^{(4)}$ and $\mathbf{\Delta}^7 = \mathbf{s}^{(5)}$. The mass matrix, stiffness matrix and force vector can be given as

$$\begin{aligned}
M_{ij}^{\alpha\beta} &= \int_{x_a}^{x_b} m_{(\alpha+\beta-2)} \frac{d^2{\phi_i}^{(\alpha)}}{dt^2} \frac{d^2{\phi_j}^{(\beta)}}{dt^2}\, dx \quad \text{for} \quad \alpha,\ \beta = 1,2,3,4. \\
M_{ij}^{\alpha\beta} &= \int_{x_a}^{x_b} m_{(\alpha+\beta-10)} \frac{d^2{\phi_i}^{(\alpha)}}{dt^2} \frac{d^2{\phi_j}^{(\beta)}}{dt^2}\, dx \quad \text{for} \quad \alpha,\ \beta = 5,6,7. \\
M_{ij}^{\alpha\beta} &= 0 \quad \text{for} \quad (\alpha = 1,2,3,4 \text{ and } \beta = 5,6,7) \quad \text{or} \quad (\alpha = 5,6,7 \text{ and } \beta = 1,2,3,4) \\
K_{ij}^{\alpha\beta} &= \int_{x_a}^{x_b} \Bigg[\left(A_{11}^{(\alpha+\beta-2)} + \frac{(\alpha-1)(\beta-1)}{8} S_{11}^{(\alpha+\beta-4)} \right) \frac{d{\phi_i}^{(\alpha)}}{dx} \frac{d{\phi_j}^{(\beta)}}{dx} \\
&\qquad + (\alpha-1)(\beta-1) \left(B_{11}^{(\alpha+\beta-4)} + \frac{1}{8}(\alpha-2)(\beta-2) S_{11}^{(\alpha+\beta-6)} \right) {\phi_i}^{(\alpha)} {\phi_j}^{(\beta)} \Bigg] dx \\
&\qquad\qquad \text{for } \alpha,\ \text{and } \beta = 1,2,3,4. \\
K_{ij}^{\alpha\beta} &= \int_{x_a}^{x_b} \Bigg[(\beta-5) A_{12}^{(\alpha+\beta-7)} \frac{d{\phi_i}^{(\alpha)}}{dx} {\phi_j}^{(\beta)} - \frac{1}{8}(\alpha-1) S_{11}^{(\alpha+\beta-7)} \frac{d{\phi_i}^{(\alpha)}}{dx} \frac{d^2{\phi_j}^{(\beta)}}{dx^2} \\
&\qquad + (\alpha-1) \left(B_{11}^{(\alpha+\beta-7)} - \frac{1}{8}(\alpha-2)(\beta-5) S_{11}^{(\alpha+\beta-9)} \right) {\phi_i}^{(\alpha)} \frac{d{\phi_j}^{(\beta)}}{dx} \Bigg] dx \\
&\qquad\qquad \text{for } \alpha = 1,2,3,4. \text{ and } \beta = 6,7. \\
K_{ij}^{\alpha\beta} &= \int_{x_a}^{x_b} \Bigg[(\alpha-5) A_{12}^{(\alpha+\beta-7)} {\phi_i}^{(\alpha)} \frac{d{\phi_i}^{(\beta)}}{dx} \\
&\qquad + (\beta-1) \left(B_{11}^{(\alpha+\beta-7)} - \frac{1}{8}(\alpha-5)(\beta-2) S_{11}^{(\alpha+\beta-9)} \right) \frac{d{\phi_i}^{(\alpha)}}{dx} {\phi_j}^{(\beta)} \\
&\qquad - \frac{1}{8}(\beta-1) S_{11}^{(\alpha+\beta-7)} \frac{d^2{\phi_i}^{(\alpha)}}{dx^2} \frac{d{\phi_j}^{(\beta)}}{dx} \Bigg] dx, \quad \text{for } \alpha = 6,7 \text{ and } \beta = 1,2,3,4.
\end{aligned}$$

$$
\begin{aligned}
K_{ij}^{\alpha\beta} &= \int_{x_a}^{x_b} \Big[(\beta-5)(\alpha-5)A_{11}^{(\alpha+\beta-12)}\phi_i{}^{(\alpha)}\phi_j{}^{(\beta)} \\
&\quad + \left(\frac{1}{8}(\beta-5)(\alpha-5)S_{11}^{(\alpha+\beta-12)} + B_{11}^{(\alpha+\beta-10)}\right)\frac{d\phi_i{}^{(\alpha)}}{dx}\frac{d\phi_j{}^{(\beta)}}{dx} \\
&\quad + \frac{1}{8}S_{11}^{(\alpha+\beta-10)}\frac{d^2\phi_i{}^{(\alpha)}}{dx^2}\frac{d^2\phi_j{}^{(\beta)}}{dx^2}\Big]\,dx, \qquad \text{for } \alpha \text{ and } \beta = 6,7. \\
K_{ij}^{\alpha 5} &= \int_{x_a}^{x_b} \Big[\frac{1}{2}A_{11}^{(\alpha-1)}\frac{dw}{dx}\frac{d\phi_i{}^{(\alpha)}}{dx}\frac{d\phi_j{}^{(5)}}{dx} + (\alpha-1)B_{11}^{(\alpha-2)}\phi_i{}^{(\alpha)}\frac{d\phi_j{}^{(5)}}{dx} \\
&\quad - \frac{1}{8}(\alpha-1)S_{11}^{(\alpha-2)}\frac{d\phi_i{}^{(\alpha)}}{dx}\frac{d^2\phi_j{}^{(5)}}{dx^2}\Big]\,dx, \qquad \text{for } \alpha = 1,2,3,4 \\
K_{ij}^{55} &= \int_{x_a}^{x_b} \Big[\frac{1}{2}A_{11}^{(0)}(\frac{dw}{dx})^2\frac{d\phi_i{}^{(5)}}{dx}\frac{d\phi_j{}^{(5)}}{dx} + B_{11}^{(0)}\frac{d\phi_i{}^{(5)}}{dx}\frac{d\phi_j{}^{(5)}}{dx} \\
&\quad + \frac{1}{8}S_{11}^{(0)}\frac{d^2\phi_i{}^{(5)}}{dx^2}\frac{d^2\phi_j{}^{(5)}}{dx^2}\Big]\,dx \\
K_{ij}^{\alpha 5} &= \int_{x_a}^{x_b} \Big[\frac{1}{2}(\alpha-5)A_{12}^{(\alpha-6)}\frac{dw}{dx}\phi_i{}^{(\alpha)}\frac{d\phi_j{}^{(5)}}{dx} + B_{11}^{(\alpha-5)}\frac{d\phi_i{}^{(\alpha)}}{dx}\frac{d\phi_j{}^{(5)}}{dx} \\
&\quad + \frac{1}{8}S_{11}^{(k-5)}\frac{d^2\phi_i{}^{(k)}}{d^2x}\frac{d^2\phi_j{}^{(5)}}{d^2x}\Big]\,dx, \qquad \text{for } \alpha = 6,7 \\
K_{ij}^{5\beta} &= \int_{x_a}^{x_b} \Big[A_{11}^{(\beta-1)}\frac{dw}{dx}\frac{d\phi_i{}^{(5)}}{dx}\frac{d\phi_j{}^{(\beta)}}{dx} + (\beta-1)B_{11}^{(\beta-2)}\frac{d\phi_i{}^{(5)}}{dx}\phi_j{}^{(\beta)} \\
&\quad - \frac{1}{8}(\beta-1)S_{11}^{(\beta-2)}\frac{d^2\phi_i{}^{(5)}}{dx^2}\frac{d\phi_j{}^{(\beta)}}{dx}\Big]\,dx, \qquad \text{for } \beta = 1,2,3,4. \\
K_{ij}^{5\beta} &= \int_{x_a}^{x_b} \Big[(\beta-5)A_{12}^{(\beta-6)}\frac{dw}{dx}\frac{d\phi_i{}^{(5)}}{dx}\phi_j{}^{(\beta)} + \frac{1}{8}S_{11}^{(\beta-5)}\frac{d^2\phi_i{}^{(5)}}{dx^2}\frac{d^2\phi_j{}^{(\beta)}}{dx^2} \\
&\quad + B_{11}^{(\beta-5)}\frac{d\phi_i{}^{(5)}}{dx}\frac{d\phi_j{}^{(\beta)}}{dx}\Big]\,dx, \qquad \text{for } \beta = 6,7 \qquad (6.36)
\end{aligned}
$$

$$
\begin{aligned}
F_i{}^{\alpha} &= \int_{x_a}^{x_b} \left[\frac{d\phi_i{}^{(\alpha)}}{dx}X_T^{(\alpha-1)} + \phi_i{}^{(k)}F_x^{(\alpha-1)} + \frac{1}{2}(\alpha-1)c_y^{(\alpha-2)}\phi_i{}^{(\alpha)}\right]dx, \ \text{for } \alpha = 1,2,3,4. \\
F_i{}^{5} &= \int_{x_a}^{x_b} \left[\frac{d\phi_i{}^{(5)}}{dx}\frac{dw}{dx}X_T^{(0)} + \phi_i{}^{(5)}F_z^{(0)} + \frac{1}{2}\frac{dc_y^{(0)}}{dx}\phi_i{}^{(5)}\right]dx \\
F_i{}^{\alpha} &= \int_{x_a}^{x_b} \left[\phi_i{}^{(\alpha)}F_z^{(\alpha-5)} + (k-5)\phi_i{}^{(\alpha)}Z_T^{(\alpha-6)} + \frac{1}{2}c_y^{(\alpha-5)}\phi_i{}^{(\alpha)}\right]dx, \ \text{for } \alpha = 5,7. \quad (6.37)
\end{aligned}
$$

By applying time approximation, nonlinear algebraic equation of the form (6.6) can be obtained, and Newton iterative procedure can be applied for solving the nonlinear finite element equation:

$$\begin{bmatrix} \hat{\mathbf{T}}^{11} & \hat{\mathbf{T}}^{12} & \dots & \hat{\mathbf{T}}^{17} \\ \hat{\mathbf{T}}^{21} & \hat{\mathbf{T}}^{22} & \dots & \hat{\mathbf{T}}^{27} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\mathbf{T}}^{71} & \hat{\mathbf{T}}^{72} & \dots & \hat{\mathbf{T}}^{77} \end{bmatrix} \left\{ \begin{array}{c} \delta\boldsymbol{\Delta}^1 \\ \delta\boldsymbol{\Delta}^2 \\ \vdots \\ \delta\boldsymbol{\Delta}^7 \end{array} \right\}^{(r+1)}_{(s+1)} = \left\{ \begin{array}{c} \mathbf{R}^1 \\ \mathbf{R}^2 \\ \vdots \\ \mathbf{R}^7 \end{array} \right\}^{(r)} \tag{6.38}$$

where, $\hat{\mathbf{T}}$ is the tangent matrix given as

$$\hat{T}_{ij}^{\alpha\beta} = \frac{\partial R_i{}^{\alpha}}{\partial \Delta_j{}^{\beta}}, \qquad \text{for} \qquad \alpha,\ \beta = 1,2,3,4,5,6,7. \tag{6.39}$$

and

$$\begin{aligned} \hat{R}_i^{\alpha} &= \sum_{\gamma=1}^{7}\sum_{k=1}^{n_\gamma} \hat{K}_{ik}^{\alpha\gamma}\Delta_k^{\gamma} - \hat{F}_i^{\alpha} \\ &= \sum_{k=1}^{n_1} \hat{K}_{ik}^{\alpha 1} u_k + \sum_{k=1}^{n_2} \hat{K}_{ik}^{\alpha 2} s_k^{(1)} + \sum_{k=1}^{n_3} \hat{K}_{ik}^{\alpha 3} s_k^{(2)} + \sum_{k=1}^{n_4} \hat{K}_{ik}^{\alpha 4} s_k^{(3)} \\ &\quad + \sum_{k=1}^{n_5} \hat{K}_{ik}^{\alpha 5} w_k + \sum_{k=1}^{n_6} \hat{K}_{ik}^{\alpha 6} s_k^{(4)} + \sum_{k=1}^{n_7} \hat{K}_{ik}^{\alpha 7} s_k^{(5)} - \hat{F}_i^{\alpha} \end{aligned} \tag{6.40}$$

The component of tangent matrix can be given as

$$\begin{aligned} \hat{T}_{ij}^{\alpha\beta} &= \hat{K}_{ij}^{\alpha\beta}, \qquad \text{for } \alpha,\ \beta = 1,2,3,4,6,7 \\ \hat{T}_{ij}^{5\beta} &= \hat{K}_{ij}^{5\beta}, \qquad \text{for } \beta = 1,2,3,4,6,7 \\ \hat{T}_{ij}^{\alpha 5} &= \hat{K}_{ij}^{\alpha 5} + \int_{x_a}^{x_b} \frac{1}{2} A_{11}^{(\alpha-1)} \frac{dw}{dx} \frac{d\phi_i{}^{(\alpha)}}{dx} \frac{d\phi_j{}^{(5)}}{dx}\ dx, \quad \text{for } \alpha = 1,2,3,4. \\ \hat{T}_{ij}^{\alpha 5} &= \hat{K}_{ij}^{\alpha 5} + \int_{x_a}^{x_b} \frac{1}{2}(\alpha-5) A_{12}^{(\alpha-6)} \frac{dw}{dx} \phi_i{}^{(\alpha)} \frac{d\phi_j{}^{(5)}}{dx}\ dx, \quad \text{for } \alpha = 6,7. \\ \hat{T}_{ij}^{55} &= \hat{K}_{ij}^{55} + \int_{x_a}^{x_b} \left[A_{11}^{(0)} \frac{du}{dx} + A_{11}^{(1)} \frac{d\theta_x}{dx} + A_{11}^{(2)} \frac{d\phi_x}{dx} + A_{11}^{(3)} \frac{d\psi_x}{dx} \right. \\ &\quad \left. + A_{12}^{(0)}\theta_z + 2A_{12}^{(1)}\phi_z + A_{11}^{(0)} \left(\frac{dw}{dx}\right)^2 - X_T{}^{(0)} \right] \frac{d\phi_i{}^{(5)}}{dx} \frac{d\phi_j{}^{(5)}}{dx} dx \end{aligned} \tag{6.41}$$

6.4. Energy Equation

Here we construct the finite element model of Eq. (2.3). The weak form of Eq. (2.3) over a typical element through the beam thickness, $\Omega^e = (z_a, z_b)$, is given by

$$0 = \int_{z_a}^{z_b} \left[\rho c_v \delta T \frac{\partial T}{\partial t} + k(z,T) \frac{d\delta T}{dz} \frac{dT}{dz} \right] dz - Q_1 \delta T(z_a) - Q_2 \delta T(z_b) \tag{6.42}$$

where

$$Q_1 = \left[-k \frac{dT}{dz} \right]_{z_a}, \quad Q_2 = \left[k \frac{dT}{dz} \right]_{z_b} \tag{6.43}$$

Assuming approximation of T as

$$T(z) \approx \sum_{j=1}^{n} T_j(t) \psi_j(z) \tag{6.44}$$

we obtain the finite element model

$$\mathbf{C}\dot{\mathbf{T}} + \mathbf{K}(\mathbf{T})\mathbf{T} = \mathbf{Q} \tag{6.45}$$

with

$$C_{ij} = \int_{z_a}^{z_b} \rho c_v \psi_i \psi_j \, dz, \quad K_{ij} = \int_{z_a}^{z_b} k(z,T) \frac{d\psi_i}{dz} \frac{d\psi_j}{dz} \, dz, \quad Q_i = Q_1 \psi_i(z_a) + Q_2 \psi_i(z_b) \tag{6.46}$$

and

$$k(z,T) = \left[k_c(T) - k_m(T) \right] f(z) + k_m(T) \tag{6.47}$$

The fully discretized model is given by

$$\hat{\mathbf{K}}(\mathbf{T}_{s+1})\mathbf{T}_{s+1} = \hat{\mathbf{Q}} \tag{6.48}$$

where

$$\begin{gathered} \hat{\mathbf{K}}(\mathbf{T}_{s+1}) = \mathbf{C} + a_1 \mathbf{K}(\mathbf{T}_{s+1}), \quad \bar{\mathbf{K}}(\mathbf{T}_s) = \mathbf{C} - a_2 \mathbf{K}(\mathbf{T}_s) \\ \hat{\mathbf{Q}}_{s,s+1} = a_1 \mathbf{F}_{s+1} + a_2 \mathbf{F}_s + \bar{\mathbf{K}}(\mathbf{T}_s)\mathbf{T}_s, \quad a_1 = \alpha \Delta t, \quad a_2 = (1-\alpha)\Delta t \end{gathered} \tag{6.49}$$

Application of the Newton's scheme to Eq. (6.48) yields

$$\hat{\mathbf{K}}^{\tan}(\mathbf{T}_{s+1}^r)\Delta \mathbf{T}_{s+1}^{r+1} = -\mathbf{R}^r = \hat{\mathbf{Q}} - \hat{\mathbf{K}}(\mathbf{T}_{s+1}^r)\mathbf{T}_{s+1}^r \tag{6.50}$$

where the tangent matrix coefficients are given by

$$\hat{K}_{ij}^{\tan} = \hat{K}_{ij} + \int_{z_a}^{z_b} F(z,T)\frac{dT}{dz}\frac{d\psi_i}{dz}\psi_j \, dz \tag{6.51}$$

and $F(z,T)$ is given by

$$\begin{aligned} F(z,T) &= \frac{dk}{dT} = \left[k_c'(T) - k_m'(T)\right] f(z) + k_m'(T) \\ k_\alpha'(T) &= c_0\left(-2c_{-1}T^{-2} + c_1 + 2c_2T + 3c_3T^2\right), \quad \alpha = c \text{ or } m \end{aligned} \tag{6.52}$$

We note that the tangent matrix is *not* symmetric.

7. ANALYTICAL SOLUTION:GENERAL THIRD-ORDER BEAM THEORY

In the present section, analytical solution for a simply supported beam has been calculated for a general third-order beam theory neglecting geometric nonlinearity. The equations of equilibrium for linear static case are

$$0 = -\left[A_{11}^{(0)}\frac{d^2u}{dx^2} + A_{11}^{(1)}\frac{d^2\theta_x}{dx^2} + A_{11}^{(2)}\frac{d^2\phi_x}{dx^2} + A_{11}^{(3)}\frac{d^2\psi_x}{dx^2} + A_{12}^{(0)}\frac{d\theta_z}{dx} + A_{12}^{(1)}2\frac{d\phi_z}{dx}\right] \tag{7.1a}$$

$$\begin{aligned} 0 = & -\left[A_{11}^{(1)}\frac{d^2u}{dx^2} + A_{11}^{(2)}\frac{d^2\theta_x}{dx^2} + A_{11}^{(3)}\frac{d^2\phi_x}{dx^2} + A_{11}^{(4)}\frac{d^2\psi_x}{dx^2} + A_{12}^{(1)}\frac{d\theta_z}{dx} + A_{12}^{(2)}2\frac{d\phi_z}{dx}\right] \\ & + \left[B_{11}^{(0)}\left(\theta_x + \frac{dw}{dx}\right) + B_{11}^{(1)}\left(2\phi_x + \frac{d\theta_z}{dx}\right) + B_{11}^{(2)}\left(3\psi_x + \frac{d\phi_z}{dx}\right)\right] - \left[\frac{1}{8}S_{11}^{(0)}\left(\frac{d^2\theta_x}{dx^2}\right.\right. \\ & \left.\left. - \frac{d^3w}{dx^3}\right) + \frac{1}{8}S_{11}^{(1)}\left(2\frac{d^2\phi_x}{dx^2} - \frac{d^3\theta_z}{dx^3}\right) + \frac{1}{8}S_{11}^{(2)}\left(3\frac{d^2\psi_x}{dx^2} - \frac{d^3\phi_z}{dx^3}\right)\right] \end{aligned} \tag{7.1b}$$

$$\begin{aligned} 0 = & -\left[A_{11}^{(2)}\frac{d^2u}{dx^2} + A_{11}^{(3)}\frac{d^2\theta_x}{dx^2} + A_{11}^{(4)}\frac{d^2\phi_x}{dx^2} + A_{11}^{(5)}\frac{d^2\psi_x}{dx^2} + A_{12}^{(2)}\frac{d\theta_z}{dx} + 2A_{12}^{(3)}\frac{d\phi_z}{dx}\right] \\ & +2\left[B_{11}^{(1)}\left(\theta_x + \frac{dw}{dx}\right) + B_{11}^{(2)}\left(2\phi_x + \frac{d\theta_z}{dx}\right) + B_{11}^{(3)}\left(3\psi_x + \frac{d\phi_z}{dx}\right)\right] \\ & -\left[\frac{1}{4}S_{11}^{(1)}\left(\frac{d^2\theta_x}{dx^2} - \frac{d^3w}{dx^3}\right) + \frac{1}{4}S_{11}^{(2)}\left(2\frac{d^2\phi_x}{dx^2} - \frac{d^3\theta_z}{dx^3}\right) + \frac{1}{4}S_{11}^{(3)}\left(3\frac{d^2\psi_x}{dx^2} - \frac{d^3\phi_z}{dx^3}\right)\right] \\ & +\left[\frac{1}{4}S_{11}^{(0)}\left(2\phi_x - \frac{d\theta_z}{dx}\right) + \frac{1}{4}S_{11}^{(1)}\left(6\psi_x - 2\frac{d\phi_z}{dx}\right)\right] \end{aligned} \tag{7.1c}$$

$$\begin{aligned} 0 = & -\left[A_{11}^{(3)}\frac{d^2u}{dx^2} + A_{11}^{(4)}\frac{d^2\theta_x}{dx^2} + A_{11}^{(5)}\frac{d^2\phi_x}{dx^2} + A_{11}^{(6)}\frac{d^2\psi_x}{dx^2} + A_{12}^{(3)}\frac{d\theta_z}{dx} + 2A_{12}^{(4)}\frac{d\phi_z}{dx}\right] \\ & +3\left[B_{11}^{(2)}\left(\theta_x + \frac{dw}{dx}\right) + B_{11}^{(3)}\left(2\phi_x + \frac{d\theta_z}{dx}\right) + B_{11}^{(4)}\left(3\psi_x + \frac{d\phi_z}{dx}\right)\right] \\ & -\frac{3}{2}\left[\frac{1}{4}S_{11}^{(2)}\left(\frac{d^2\theta_x}{dx^2} - \frac{d^3w}{dx^3}\right) + \frac{1}{4}S_{11}^{(3)}\left(2\frac{d^2\phi_x}{dx^2} - \frac{d^3\theta_z}{dx^3}\right) + \frac{1}{4}S_{11}^{(4)}\left(3\frac{d^2\psi_x}{dx^2} - \frac{d^3\phi_z}{dx^3}\right)\right] \\ & +3\left[\frac{1}{4}S_{11}^{(1)}\left(2\phi_x - \frac{d\theta_z}{dx}\right) + \frac{1}{4}S_{11}^{(2)}\left(6\psi_x - 2\frac{d\phi_z}{dx}\right)\right] \end{aligned} \tag{7.1d}$$

$$\begin{aligned} 0 = & -\left[B_{11}^{(0)}\left(\frac{d\theta_x}{dx} + \frac{d^2w}{dx^2}\right) + B_{11}^{(1)}\left(2\frac{d\phi_x}{dx} + \frac{d^2\theta_z}{dx^2}\right) + B_{11}^{(2)}\left(3\frac{d\psi_x}{dx} + \frac{d^2\phi_z}{dx^2}\right)\right] - F_z \\ & -\left[\frac{1}{8}S_{11}^{(0)}\left(\frac{d^3\theta_x}{dx^3} - \frac{d^4w}{dx^4}\right) + \frac{1}{8}S_{11}^{(1)}\left(2\frac{d^3\phi_x}{dx^3} - \frac{d^4\theta_z}{dx^4}\right) + \frac{1}{8}S_{11}^{(2)}\left(3\frac{d^3\psi_x}{dx^3} - \frac{d^4\phi_z}{dx^4}\right)\right] \end{aligned} \tag{7.1e}$$

$$0 = -\left[B_{11}^{(1)}\left(\frac{d\theta_x}{dx} + \frac{d^2w}{dx^2}\right) + B_{11}^{(2)}\left(2\frac{d\phi_x}{dx} + \frac{d^2\theta_z}{dx^2}\right) + B_{11}^{(3)}\left(3\frac{d\psi_x}{dx} + \frac{d^2\phi_z}{dx^2}\right)\right]$$

$$
\begin{aligned}
&+\left[A_{12}^{(0)}\frac{du}{dx}+A_{12}^{(1)}\frac{d\theta_x}{dx}+A_{12}^{(2)}\frac{d\phi_x}{dx}+A_{12}^{(3)}\frac{d\psi_x}{dx}+A_{11}^{(0)}\theta_z+A_{11}^{(1)}2\phi_z\right]-F_z^{(1)}\\
&-\frac{1}{2}\left[\frac{1}{4}S_{11}^{(1)}\left(\frac{d^3\theta_x}{dx^3}-\frac{d^4w}{dx^4}\right)+\frac{1}{4}S_{11}^{(2)}\left(2\frac{d^3\phi_x}{dx^3}-\frac{d^4\theta_z}{dx^4}\right)+\frac{1}{4}S_{11}^{(3)}\left(3\frac{d^3\psi_x}{dx^3}-\frac{d^4\phi_z}{dx^4}\right)\right]\\
&+\frac{1}{2}\left[\frac{1}{4}S_{11}^{(0)}\left(2\frac{d\phi_x}{dx}-\frac{d^2\theta_z}{dx^2}\right)+\frac{1}{4}S_{11}^{(1)}\left(6\frac{d\psi_x}{dx}-2\frac{d^2\phi_z}{dx^2}\right)\right] \qquad (7.1f)
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& -\left[B_{11}^{(2)}\left(\frac{d\theta_x}{dx}+\frac{d^2w}{dx^2}\right)+B_{11}^{(3)}\left(2\frac{d\phi_x}{dx}+\frac{d^2\theta_z}{dx^2}\right)+B_{11}^{(4)}\left(3\frac{d\psi_x}{dx}+\frac{d^2\phi_z}{dx^2}\right)\right]\\
&+2\left[A_{12}^{(1)}\frac{du}{dx}+A_{12}^{(2)}\frac{d\theta_x}{dx}+A_{12}^{(3)}\frac{d\phi_x}{dx}+A_{12}^{(4)}\frac{d\psi_x}{dx}+A_{11}^{(1)}\theta_z+A_{11}^{(2)}2\phi_z\right]-F_z^{(2)}\\
&-\frac{1}{2}\left[\frac{1}{4}S_{11}^{(2)}\left(\frac{d^3\theta_x}{dx^3}-\frac{d^4w}{dx^4}\right)+\frac{1}{4}S_{11}^{(3)}\left(2\frac{d^3\phi_x}{dx^3}-\frac{d^4\theta_z}{dx^4}\right)+\frac{1}{4}S_{11}^{(4)}\left(3\frac{d^3\psi_x}{dx^3}-\frac{d^4\phi_z}{dx^4}\right)\right]\\
&+\left[\frac{1}{4}S_{11}^{(1)}\left(2\frac{d\phi_x}{dx}-\frac{d^2\theta_z}{dx^2}\right)+\frac{1}{4}S_{11}^{(2)}\left(6\frac{d\psi_x}{dx}-2\frac{d^2\phi_z}{dx^2}\right)\right] \qquad (7.1g)
\end{aligned}
$$

The static bending solution can be assumed of the following form,

$$
\begin{aligned}
u(x) &= \sum_{n=1}^{\infty} U_n \cos\frac{n\pi x}{L}, \quad \theta_x(x)=\sum_{n=1}^{\infty} S_n^1\cos\frac{n\pi x}{L}, \quad \phi_x(x)=\sum_{n=1}^{\infty} S_n^2\cos\frac{n\pi x}{L}\\
\psi_x(x) &= \sum_{n=1}^{\infty} S_n^3 \cos\frac{n\pi x}{L}, \quad w(x)=\sum_{n=1}^{\infty} W_n\sin\frac{n\pi x}{L}, \quad \theta_z(x)=\sum_{n=1}^{\infty} S_n^4\sin\frac{n\pi x}{L}\\
\phi_z(x) &= \sum_{n=1}^{\infty} S_n^5 \sin\frac{n\pi x}{L} \qquad (7.2)
\end{aligned}
$$

and the applied force is expressed as

$$
f(x)=\sum_{n=1}^{\infty} F_n\sin\frac{n\pi x}{L} \qquad (7.3)
$$

Substituting $u, \theta_x, \phi_x, \psi_x, w, \theta_z$, and ϕ_z from Eq. (7.2) in the linear equations of motion (7.1a)–(7.1f), following equations are obtained:

$$
\begin{aligned}
0 \;=\;& \sum_{n=1}^{\infty}\frac{n\pi}{L}\Big[A_{11}^{(0)}\frac{n\pi}{L}U_n+A_{11}^{(1)}\frac{n\pi}{L}S_n^1+A_{11}^{(2)}\frac{n\pi}{L}\;S_n^2+A_{11}^{(3)}\frac{n\pi}{L}S_n^3\\
&\quad -A_{12}^{(0)}S_n^4-2A_{12}^{(1)}S_n^5\Big]\cos\frac{n\pi x}{L} \qquad (7.4a)
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& \sum_{n=1}^{\infty}\Big[\frac{n\pi}{L}\left\{A_{11}^{(1)}\frac{n\pi}{L}U_n+A_{11}^{(2)}\frac{n\pi}{L}S_n^1+A_{11}^{(3)}\frac{n\pi}{L}S_n^2+A_{11}^{(4)}\frac{n\pi}{L}S_n^3-A_{12}^{(1)}S_n^4-2A_{12}^{(2)}S_n^5\right\}\\
&\quad +\left\{B_{11}^{(0)}\left(S_n^1+\frac{n\pi}{L}W_n\right)+B_{11}^{(1)}\left(2S_n^2+\frac{n\pi}{L}S_n^4\right)+B_{11}^{(2)}\left(3S_n^3+\frac{n\pi}{L}S_n^5\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{8}\left(\frac{n\pi}{L}\right)^2\left\{S_{11}^{(0)}\left(S_n^1-\frac{n\pi}{L}W_n\right)+S_{11}^{(1)}\left(2S_n^2-\frac{n\pi}{L}S_n^4\right)\right.\\
&\left.\left.+S_{11}^{(2)}\left(3S_n^3-\frac{n\pi}{L}S_n^5\right)\right\}\right]\cos\frac{n\pi x}{L} \qquad (7.4b)
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; &\sum_{n=1}^{\infty}\left[\frac{n\pi}{L}\left\{A_{11}^{(2)}\frac{n\pi}{L}U_n+A_{11}^{(3)}\frac{n\pi}{L}S_n^1+A_{11}^{(4)}\frac{n\pi}{L}S_n^2+A_{11}^{(5)}\frac{n\pi}{L}S_n^3-A_{12}^{(2)}S_n^4-2A_{12}^{(3)}S_n^5\right\}\right.\\
&+2\left\{B_{11}^{(1)}\left(S_n^1+\frac{n\pi}{L}W_n\right)+B_{11}^{(2)}\left(2S_n^2+\frac{n\pi}{L}S_n^4\right)+B_{11}^{(3)}\left(3S_n^3+\frac{n\pi}{L}S_n^5\right)\right\}\\
&+\frac{1}{4}\left(\frac{n\pi}{L}\right)^2\left\{S_{11}^{(1)}\left(S_n^1-\frac{n\pi}{L}W_n\right)+S_{11}^{(2)}\left(2S_n^2-\frac{n\pi}{L}S_n^4\right)+S_{11}^{(3)}\left(3S_n^3-\frac{n\pi}{L}S_n^5\right)\right\}\\
&\left.+\frac{1}{4}\left\{S_{11}^{(0)}\left(2S_n^2-\frac{n\pi}{L}S_n^4\right)+S_{11}^{(1)}\left(6S_n^3-2\frac{n\pi}{L}S_n^5\right)\right\}\right]\cos\frac{n\pi x}{L} \qquad (7.4c)
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; &\sum_{n=1}^{\infty}\left[\frac{n\pi}{L}\left\{A_{11}^{(3)}\frac{n\pi}{L}U_n+A_{11}^{(4)}\frac{n\pi}{L}S_n^1+A_{11}^{(5)}\frac{n\pi}{L}S_n^2+A_{11}^{(6)}\frac{n\pi}{L}S_n^3-A_{12}^{(3)}S_n^4-2A_{12}^{(4)}S_n^5\right\}\right.\\
&+3\left(B_{11}^{(2)}\left(S_n^1+\frac{n\pi}{L}W_n\right)+B_{11}^{(3)}\left(2S_n^2+\frac{n\pi}{L}S_n^4\right)+B_{11}^{(4)}\left(3S_n^3+\frac{n\pi}{L}S_n^5\right)\right)\\
&+\frac{3}{8}\left(\frac{n\pi}{L}\right)^2\left\{S_{11}^{(2)}\left(S_n^1-\frac{n\pi}{L}W_n\right)+S_{11}^{(3)}\left(2S_n^2-\frac{n\pi}{L}S_n^5\right)+S_{11}^{(4)}\left(3S_n^3-\frac{n\pi}{L}S_n^5\right)\right\}\\
&\left.+\frac{3}{4}\left\{S_{11}^{(1)}\left(2S_n^2-\frac{n\pi}{L}S_n^4\right)+S_{11}^{(2)}\left(6S_n^3-2\frac{n\pi}{L}S_n^5\right)\right\}\right]\cos\frac{n\pi x}{L} \qquad (7.4d)
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; &\sum_{n=1}^{\infty}\left[\frac{n\pi}{L}\left\{B_{11}^{(0)}\left(S_n^1+\frac{n\pi}{L}W_n\right)+B_{11}^{(1)}\left(2S_n^2+\frac{n\pi}{L}S_n^4\right)+B_{11}^{(2)}\left(3S_n^1+\frac{n\pi}{L}S_n^5\right)\right\}\right.\\
&-(F_z)_n-\frac{1}{8}\left(\frac{n\pi}{L}\right)^3\left\{S_{11}^{(0)}\left(S_n^1-\frac{n\pi}{L}W_n\right)+S_{11}^{(1)}\left(2S_n^2-\frac{n\pi}{L}S_n^4\right)\right.\\
&\left.\left.+S_{11}^{(2)}\left(3S_n^3-\frac{n\pi}{L}S_n^5\right)\right\}\right]\sin\frac{n\pi x}{L} \qquad (7.4e)
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; &\sum_{n=1}^{\infty}\left[\frac{n\pi}{L}\left\{B_{11}^{(1)}\left(S_n^1+\frac{n\pi}{L}W_n\right)+B_{11}^{(2)}\left(2S_n^2+\frac{n\pi}{L}S_n^4\right)+B_{11}^{(3)}\left(3S_n^3+\frac{n\pi}{L}S_n^5\right)\right\}\right.\\
&-\left\{A_{12}^{(0)}\frac{n\pi}{L}U_n+A_{12}^{(1)}\frac{n\pi}{L}S_n^1+A_{12}^{(2)}\frac{n\pi}{L}S_n^2+A_{12}^{(3)}\frac{n\pi}{L}S_n^3-A_{11}^{(0)}S_n^4-A_{11}^{(1)}2S_n^5\right\}\\
&-\frac{1}{8}\left(\frac{n\pi}{L}\right)^3\left\{S_{11}^{(1)}\left(S_n^1-\frac{n\pi}{L}W_n\right)+S_{11}^{(2)}\left(2S_n^2-\frac{n\pi}{L}S_n^4\right)+S_{11}^{(3)}\left(3S_n^3-\frac{n\pi}{L}S_n^5\right)\right\}\\
&\left.-(F_z^{(1)})_n-\frac{1}{8}\frac{n\pi}{L}\left\{S_{11}^{(0)}\left(2S_n^2-\frac{n\pi}{L}S_n^4\right)+S_{11}^{(1)}\left(6S_n^3-2\frac{n\pi}{L}S_n^5\right)\right\}\right]\sin\frac{n\pi x}{L} \qquad (7.4f)
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; &\sum_{n=1}^{\infty}\left[\frac{n\pi}{L}\left\{B_{11}^{(2)}\left(S_n^1+\frac{n\pi}{L}W_n\right)+B_{11}^{(3)}\left(2S_n^2+\frac{n\pi}{L}S_n^4\right)+B_{11}^{(4)}\left(3S_n^3+\frac{n\pi}{L}S_n^5\right)\right\}\right.\\
&-2\left\{A_{12}^{(1)}\frac{n\pi}{L}U_n+A_{12}^{(2)}\frac{n\pi}{L}S_n^1+A_{12}^{(3)}\frac{n\pi}{L}S_n^2+A_{12}^{(4)}\frac{n\pi}{L}S_n^3-A_{11}^{(1)}\frac{n\pi}{L}S_n^4-A_{11}^{(2)}2\frac{n\pi}{L}S_n^5\right\}\\
&-\frac{1}{8}\left(\frac{n\pi}{L}\right)^3\left\{S_{11}^{(2)}\left(S_n^1-\frac{n\pi}{L}W_n\right)+S_{11}^{(3)}\left(2S_n^2-\frac{n\pi}{L}S_n^4\right)+S_{11}^{(4)}\left(3S_n^3-\frac{n\pi}{L}S_n^5\right)\right\}
\end{aligned}
$$

$$-(F_z^{(2)})_n - \frac{1}{4}\frac{n\pi}{L}\left\{S_{11}^{(1)}\left(2S_n^2 - \frac{n\pi}{L}S_n^4\right) + S_{11}^{(2)}\left(6S_n^3 - 2\frac{n\pi}{L}S_n^5\right)\right\}\Bigg]\sin\frac{n\pi x}{L} \quad (7.4g)$$

Eqs. (7.4a)–(7.4g) are true for every x in the domain, hence the coefficient of each sine and cosine term should be equal to zero, which would result in a set of seven simultaneous equations in terms of the nth coefficient of the degrees of freedom $(u, \theta_x, \phi_x, \psi_x, w, \theta_z, \phi_z)$, which can be represented in a matrix form as:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} & K_{17} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} & K_{27} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} & K_{37} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} & K_{47} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} & K_{57} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} & K_{67} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} \end{bmatrix} \begin{Bmatrix} U_n \\ S_n^1 \\ S_n^2 \\ S_n^3 \\ W_n \\ S_n^4 \\ S_n^5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ (F_z)_n \\ (F_z^{(1)})_n \\ (F_z^{(2)})_n \end{Bmatrix} \quad (7.5)$$

where

$$\begin{aligned}
K_{ij} &= \left(\frac{n\pi}{L}\right)^2\left(A_{11}^{(i+j-2)} + \frac{(i-1)(j-1)}{8}S_{11}^{(i+j-4)}\right) \\
&\quad +(i-1)(j-1)\left(B_{11}^{(i+j-4)} + \frac{1}{8}(i-2)(j-2)S_{11}^{(i+j-6)}\right) \\
&\qquad\qquad\qquad\qquad \text{for,} \quad i,\ j = 1,2,3,4. \\
K_{ij} &= -(j-5)\frac{n\pi}{L}A_{12}^{(i+j-7)} - \frac{1}{8}(i-1)\left(\frac{n\pi}{L}\right)^3 S_{11}^{(i+j-7)} \\
&\quad +(i-1)\frac{n\pi}{L}\left(B_{11}^{(i+j-7)} - \frac{1}{8}(i-2)(j-5)S_{11}^{(i+j-9)}\right) \\
&\qquad\qquad\qquad\qquad \text{for} \quad i = 1,2,3,4. \text{ and } j = 5,6,7. \\
K_{ij} &= -(i-5)\frac{n\pi}{L}A_{12}^{(i+j-7)} - \frac{1}{8}\left(\frac{n\pi}{L}\right)^3(j-1)S_{11}^{(i+j-7)} \\
&\quad +(j-1)\frac{n\pi}{L}\left(B_{11}^{(i+j-7)} - \frac{1}{8}(i-5)(j-2)S_{11}^{(i+j-9)}\right) \\
&\qquad\qquad\qquad\qquad \text{for} \quad i = 5,6,7. \text{ and } j = 1,2,3,4. \\
K_{ij} &= (j-5)(i-5)A_{11}^{(i+j-12)} + \frac{1}{8}\left(\frac{n\pi}{L}\right)^4 S_{11}^{(i+j-10)} \\
&\quad +\left(\frac{n\pi}{L}\right)^2\left(\frac{1}{8}(j-5)(i-5)S_{11}^{(i+j-12)} + B_{11}^{(i+j-10)}\right) \\
&\qquad\qquad\qquad\qquad \text{for} \quad i,\ j = 5,6,7.
\end{aligned} \quad (7.6)$$

and

$$(F_z^{(i)})_n = \int_A F_n z^i \, dA + \left(\frac{h}{2}\right)^i \left[(q_t)_n + (-1)^i (q_b)_n\right] \text{ for } \quad i = 0,1,2. \quad (7.7)$$

Simply supported homogeneous beam of following geometric and material parameter is considered for numerical example:

$$
\begin{aligned}
&E_2 = E_1 = E = 1.44 \text{ GPa}, \quad \nu = 0.38, \\
&h = 17.6 \times 10^{-6} \text{ m}, \quad b = 2h, \quad L = 20h,
\end{aligned} \tag{7.8}
$$

Results for non dimensional central vertical deflection ($\bar{w} = wEI/q_0L^4$) are tabulated for uniform ($q = q_0$) and sinusoidal load ($q = q_0 \sin \frac{x}{L}$) for $q_0 = 1$N/m and the general third-order beam theory (TOBT) results have been compared with EBT and TBT taken from Reddy [39] in Table 7.1 for the load acting as body force and also as the traction force on the top of the beam.

Table 7.1. Analytical solution for center deflection $\bar{w} \times 10^2$ for simply supported homogeneous beam for general third-order beam theory

	Uniform load				Sinusoidal Load			
l/h	EBT	TBT	TOBT load as body force	TOBT Traction on top	EBT	TBT	TOBT load as body force	TOBT Traction on top
0.0	1.3021	1.3103	1.3108	1.3098	1.0266	1.0333	1.0337	1.0329
0.2	1.1092	1.1162	1.1163	1.1155	0.8745	0.8802	0.8803	0.8796
0.4	0.7679	0.7731	0.7729	0.7723	0.6054	0.6096	0.6095	0.6090
0.6	0.5076	0.5116	0.5113	0.5109	0.4002	0.4034	0.4032	0.4029
0.8	0.3442	0.3475	0.3473	0.3470	0.2714	0.2741	0.2739	0.2736
1.0	0.2435	0.2464	0.2461	0.2459	0.1920	0.1943	0.1941	0.1939

8. NUMERICAL RESULTS

8.1. Micro-Structure Dependent FGM Beam

8.1.1. Pin-pin connected beam

First the homogeneous beam of following geometric and material parameters is considered:

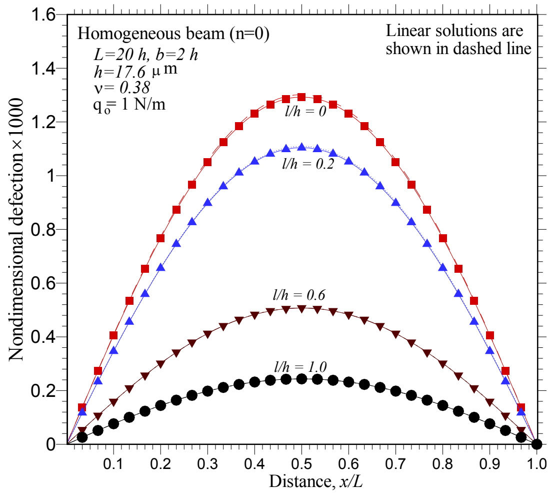
$$\begin{gathered}E_2 = E_1 = E = 1.44 \text{ GPa}, \quad \nu = 0.38, \quad K_s = \frac{5(1+\nu)}{6+5\nu} \\ h = 17.6 \times 10^{-6} \text{ m}, \quad b = 2h, \quad L = 20h\end{gathered} \tag{8.1}$$

For nonlinear finite element solution, linear and Hermite cubic interpolation function are used for u and w, respectively, in case of EBT, and quadratic elements are used for conventional TBT, i.e., for $\ell = 0$, whereas for microstructure dependent beam linear interpolation of u and ϕ_x is used and Hermite cubic approximation of w is used. For the above simply supported beam, thirty, twenty, and sixty beam elements are used for EBT and TBT ($\ell = 0$) and TBT ($\ell \neq 0$), respectively. The analytical solution (see Reddy [39]) and the linear finite element solution for the maximum vertical deflection ($\bar{w} = wEI/q_0L^4$) are compared in Table 8.1. It is noted that for

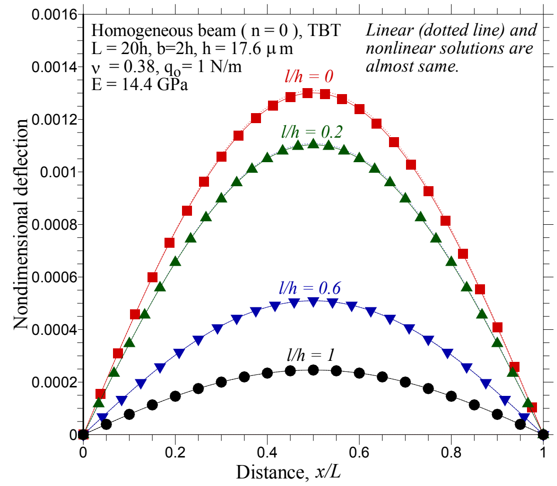
Table 8.1. Comparison of analytical and FEM(linear) solution of center deflection $\bar{w} \times 10^2$ for simply supported homogeneous beam for uniform load for EBT and TBT

l/h	method	EBT	TBT
0	Analytical	1.3021	1.3103
	FEM (linear)	1.3021	1.3103
0.6	Analytical	0.5076	0.5116
	FEM (linear)	0.5076	0.5098
1	Analytical	0.2435	0.2464
	FEM (linear)	0.2435	0.2460

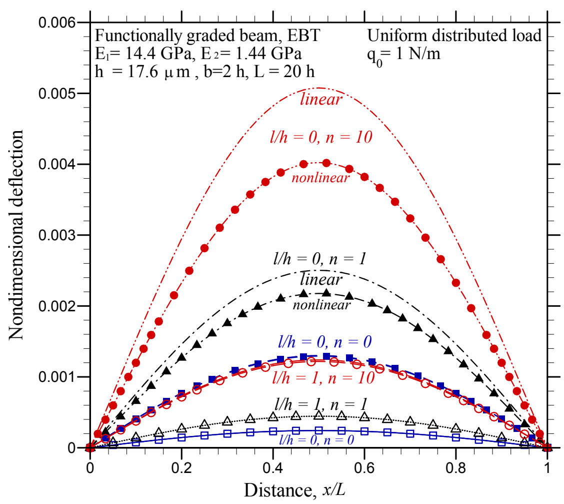
more accurate solution for microstructure dependent TBT, more number of elements or higher order elements are required. Vertical deflection ($\bar{w} = wEI/L^4$) of the beam along the length (non-dimensional, x/L) of the beam is shown for different ℓ, which is taken as a fraction of the height (h) of the beam, in Figs. 8.1(a) and 8.1(b) for both linear and nonlinear analysis under uniformly distributed load, $q_0 = 1$ N/m for both Euler-Bernoulli and Timoshenko beam theory, respectively, for homogeneous beam. The nonlinear non-dimensional transverse deflection ($\bar{w} = wEI/q_0L^4$) for functionally graded beam having power-law index, $n = 0$, $n = 1$ and $n = 10$ under uniformly distributed load are shown in Figs. 8.1(c) and 8.1(d) for $l/h = 0$ and $l/h = 1$. Linear solution of transverse deflection for the respective beams is also shown in the same figure for comparison. To see the effect of nonlinearity, maximum deflection of the FGM beam verses the transverse load applied are plotted for different power-law index of FGM in Fig. 8.2. Homogeneous and functionally graded beam of aforementioned geometric and material parameter are also analyzed considering Timoshenko beam theory.To see the shear effect, both EBT and TBT are shown together in Fig. 8.2 with respect to load applied. It is noted that for pinned–pinned connected beam, there is more nonlinearity in case of lower value of ℓ than the than higher ℓ. The shear effect in this boundary condition is not significant.



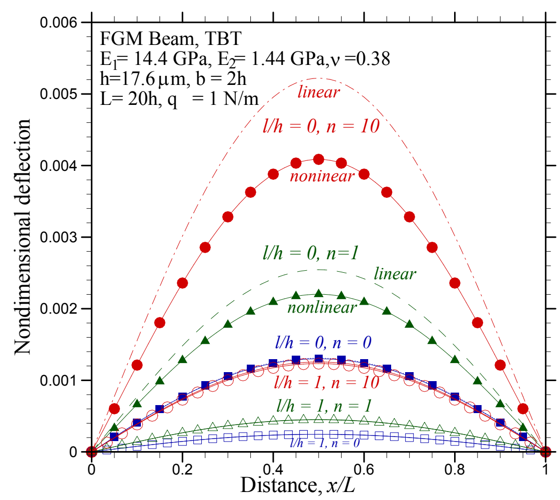
(a) homogeneous beam (EBT)



(b) homogeneous beam beam (TBT)

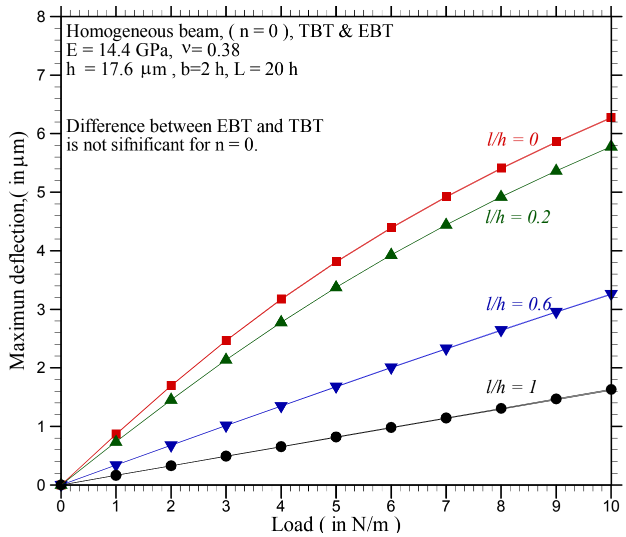


(c) Functionally graded beam (EBT)

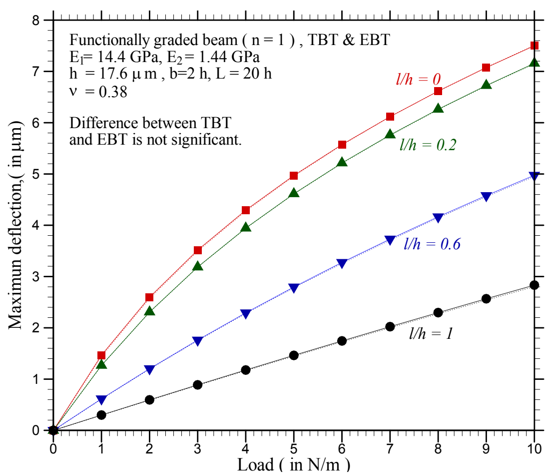


(d) Functionally graded beam (TBT)

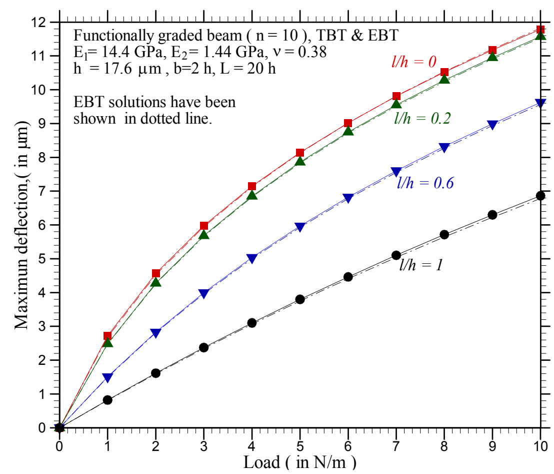
Fig. 8.1 Transverse deflection versus distance along the length of pinned-pinned connected beam



(a) homogeneous beam



(b) FG beam with $n = 1$

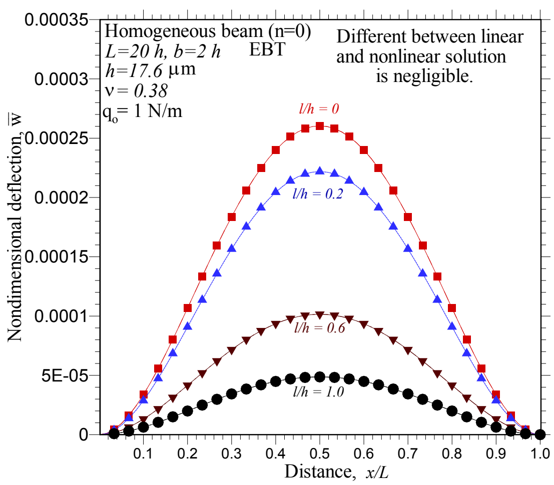


(c) FG beam with $n = 10$

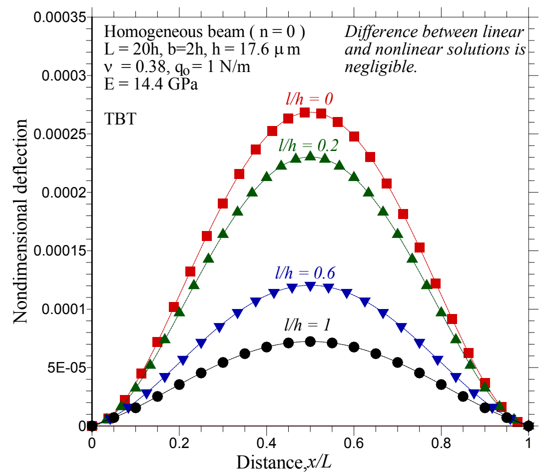
Fig. 8.2 Maximum transverse deflection versus load for pinned-pinned connected beam

8.1.2. Clamped beam

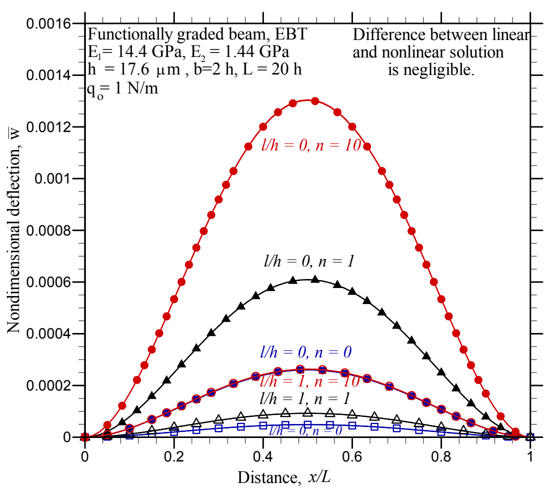
In this section, the same beam described by Eq. (8.1) with clamped boundary condition is analyzed for uniform loading condition. In Fig. 8.3, the non-dimensional transverse deflection ($\bar{w} = wEI/L^4$) along the length (non-dimensional, x/L) of the beam are plotted for different value of microstructure length parameters, ℓ for homogeneous and FGM beam considering EBT and TBT.



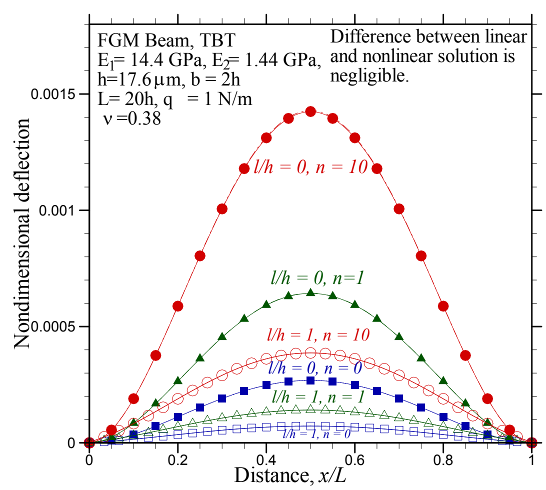
(a) homogeneous beam (EBT)



(b) homogeneous beam beam (TBT)



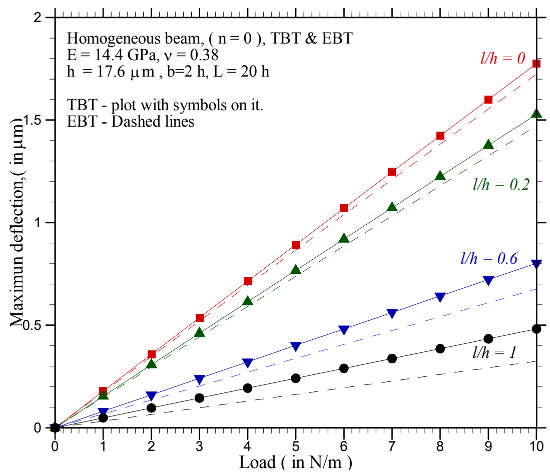
(c) Functionally graded beam (EBT)



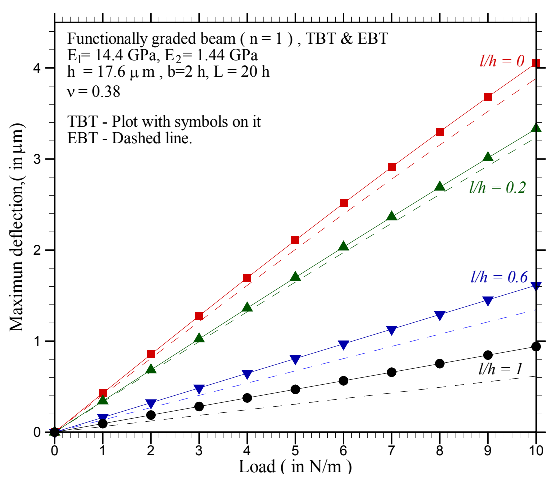
(d) Functionally graded beam (TBT)

Fig. 8.3 Transverse deflection versus distance along the length of clamped beam

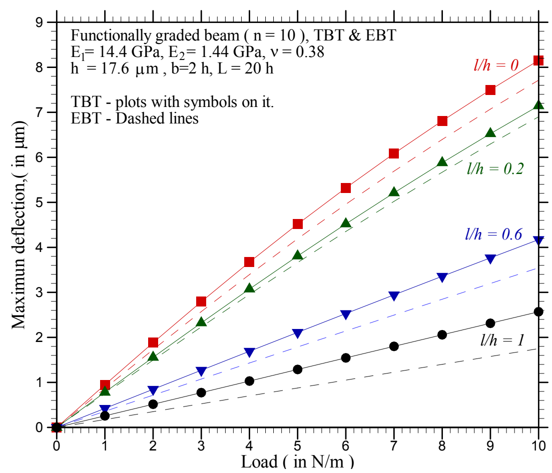
In Fig. 8.4 the maximum non-dimensional transverse deflection verses different uniform loading condition are plotted for different microstructural parameter for homogeneous and FGM beam having power-law index of $n = 1,\ 10$ considering EBT and TBT. It can be seen that the nonlinearity is less as compared to the pinned–pinned connected beam, whereas the shear effect can significantly be seen.



(a) homogeneous beam



(b) FG beam with $n = 1$



(c) FG beam with $n = 10$

Fig. 8.4 Maximum transverse deflection versus load for clamped connected beam

8.1.3. Numerical results for general third order beam theory

Simply supported beam with specifications given in Eq. (8.1) is considered; u, θ_x, ϕ_x, and ψ_x are interpolated using linear elements, whereas w, θ_z, and ϕ_z are approximated by Hermite cubic polynomials. Because of the symmetry of the problem, half domain is considered in the finite element analysis. The boundary conditions are

$$\begin{gathered} w(0)=0, \quad \theta_z(0)=0, \quad \phi_z(0)=0 \\ u(L/2)=0, \quad \theta_x(L/2)=0, \quad \phi_x(L/2)=0, \quad \psi_x(L/2)=0 \\ \frac{\partial w}{\partial x}(L/2)=0, \quad \frac{\partial \theta_z}{\partial x}(L/2)=0, \quad \frac{\partial \phi_z}{\partial x}(L/2)=0 \end{gathered} \tag{8.2}$$

Sixty two-node elements has been considered in the finite element analysis. The linear finite element solution ($\bar{w} = w \times EI/q_0L^4$) is compared with the analytical solution obtained in previous section, and they are tabulated in Table 8.2 for homogeneous beam for different ℓ/h ratio.

Table 8.2. Comparison of analytical and FEM (linear) solution of center deflection $\bar{w} \times 10^2$ for simply supported homogeneous beam for general third order beam theory.

l/h	method	Load as body force	Load as traction on top
0	Analytical	1.3108	1.3098
	FEM (linear)	1.3108	1.3098
0.2	Analytical	1.1163	1.1155
	FEM (linear)	1.1163	1.1154
0.6	Analytical	0.5113	0.5109
	FEM (linear)	0.5113	0.5109
1	Analytical	0.2461	0.2459
	FEM (linear)	0.2461	0.2459

8.2. FGM Beams under Thermo-Mechanical Loads

The temperature at bottom (metal end) surface (T_m) of the beam is taken at room temperature 25° C and at the top (ceramic end) surface (T_c) of the beam is considered as 1000° C. Analysis for different top surface temperature has also been done for some boundary conditions.

8.2.1. Temperature profile and section properties

Beam of following geometric parameters has been considered.

$$L = 2 \text{ m}, \quad h = 0.02 \text{ m}, \quad b = 2h = 0.04 \text{ m} \tag{8.3}$$

The parameter c_0, c_1, c_2, and c_3 of Eq. (2.2) for the temperature dependent properties of Zirconia and Titanium are listed in Tables 8.3 and 8.4, respectively. The

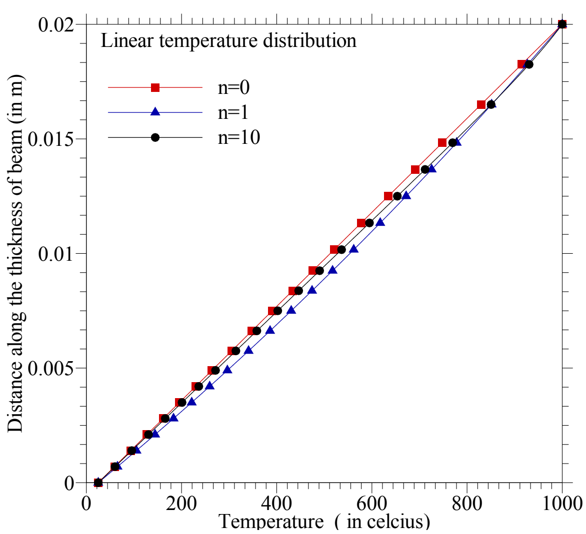
Table 8.3. Parameters for material properties of Zirconia

Property	c_0	c_{-1}	c_1 x 10^4	c_2 x 10^8	c_3 x 10^{10}
ρ, Density (kg/m^3)	5700	0	0	0	0
k, Conductivity (W/m K)	1.7	0	1.276	664.85	0
α, Coefficient of thermal expansion (K)	12.7657 x 10^{-6}	0	-14.9	0.0001	-0.06775
ν, Poisson's ratio	0.2882	0	1.13345	0	0
C_v, Specific heat (J/kg K)	487.34279	0	3.04098	-6.037232	0
E, Young's modulus (Pa)	244.26596 x 10^9	0	-13.707	121.393	-3.681378

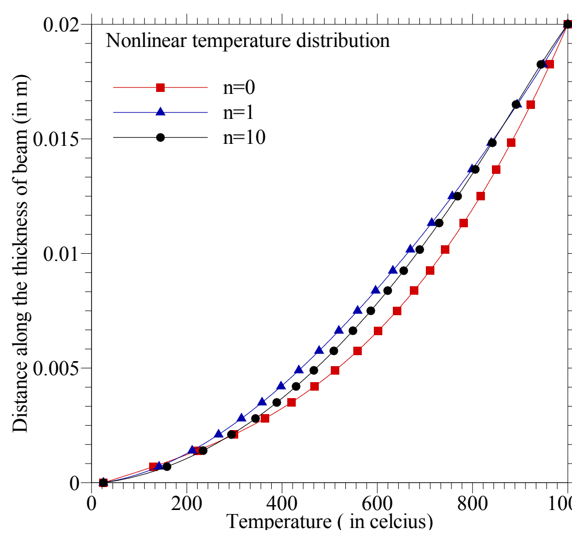
temperature profile through the beam thickness (0.02m) are shown in Fig. 8.5 for $T_c = 1000$°C and $T_m = 25$°C, for homogeneous beams and functionally graded beams having power-law index $n = 1$ and $n = 10$ considering both temperature-independent (linear) and temperature-dependent (nonlinear) thermal conductivity. The material properties like modulus of elasticity, E, and coefficient of thermal expansion, α, of

Table 8.4. Parameters for material properties of Ti6AIV

Property	c_0	c_{-1}	c_1 x 10^4	c_2 x 10^8	c_3 x 10^{10}
ρ, Density (kg/m^3)	4,429	0	0	0	0
k, Conductivity (W/m K)	1.2094	0	139.375	0	0
α, Coefficient of thermal expansion (K)	7.57876 x 10^{-6}	0	6.5 x 10^{-4}	31.467 x 10^{-8}	0
ν, Poisson's ratio	0.28838235	0	1.12136	0	0
C_v, Specific heat (J/kg K)	625.29692	0	-4.2238757	71.786536	0
E, Young's modulus (Pa)	122.55676 x 10^9	0	-4.58635	0	-3.681378

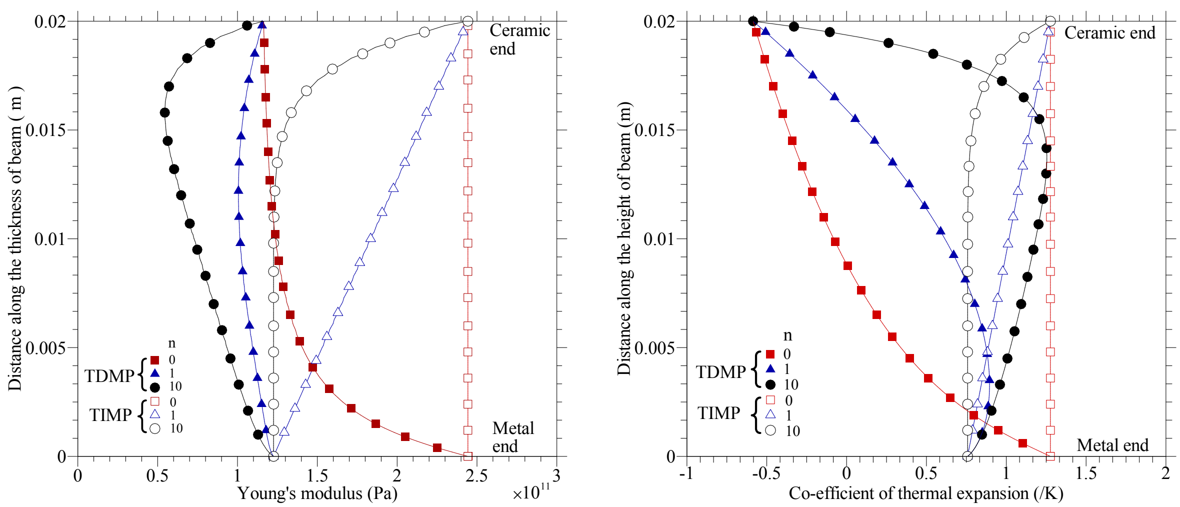


(a) Linear



(b) Nonlinear

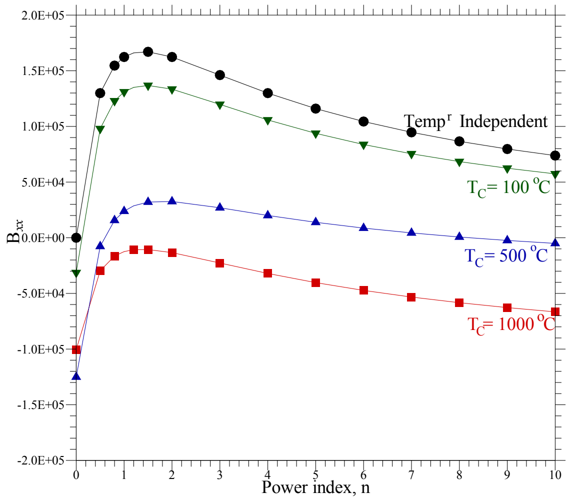
Fig. 8.5 Temperature distribution through thickness of the beam functionally graded beam as well as of homogeneous beam are shown in Figs. 8.6(a) and 8.6(b), respectively for both temperature-dependent(TDMP) and temperature-independent material properties (TIMP). Since the material properties depend on temperature, the cross section properties like bending stiffness ($D_{xx} - B_{xx}^2/A_{xx}$) and coupling stiffness B_{xx} of the functionally graded beam in turn depend on the temperature profile as well as on the power-law index of FGM beam. The variation bending stiffness and B_{xx} of the functionally graded beam verses different power-law index, n, are shown in Figs. 8.7(a) and 8.7(b), respectively for three different boundary temperatures T_c at the ceramic end of FGM beam and also for temperature independent



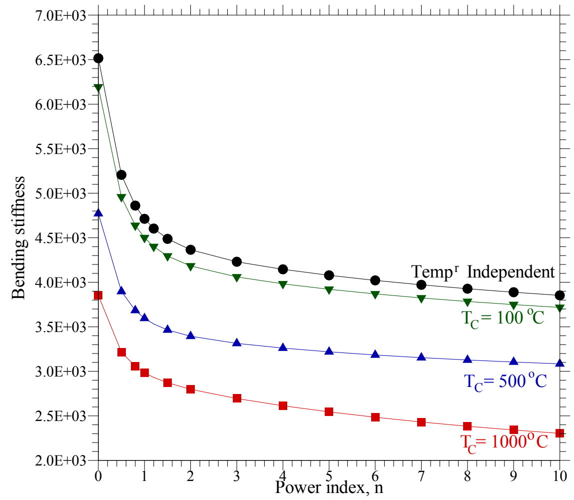
(a) Young's modulus (b) Co-efficient of thermal expansion

Fig. 8.6 Material properties through thickness of the beam

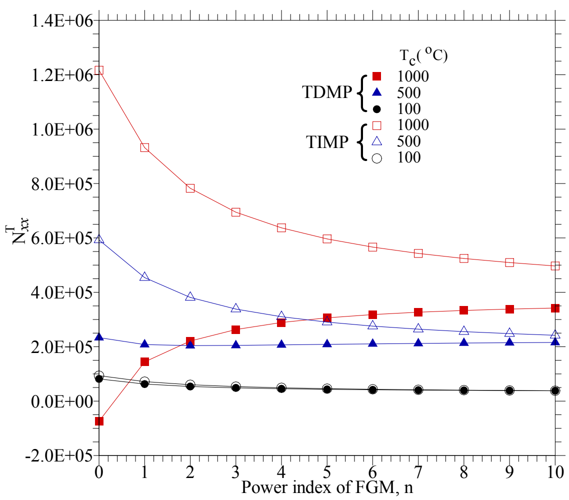
Young's modulus. Figs. 8.7 (c) and (d) show the variation of N_{xx}^T and M_{xx}^T verses power-law index n of FGM beam for different T_c considering temperature-dependent and temperature-independent material properties.



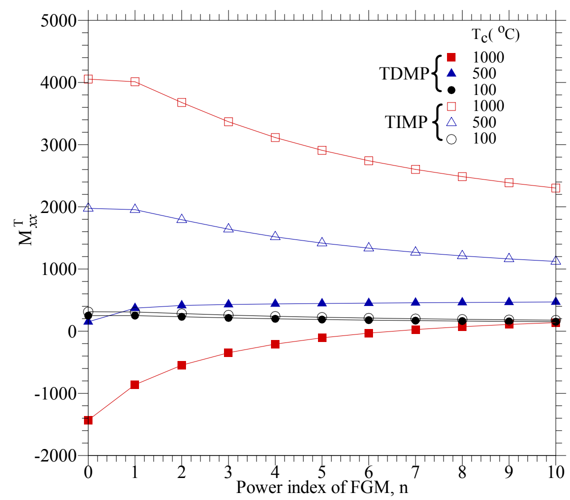
(a) B_{xx}



(b) Bending stiffness



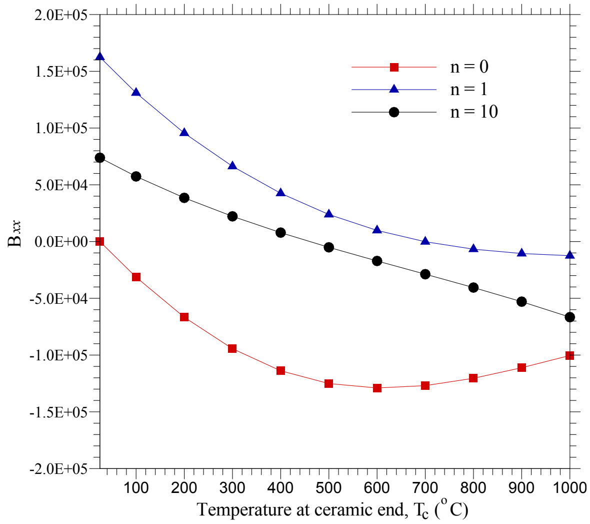
(c) N_{xx}^T



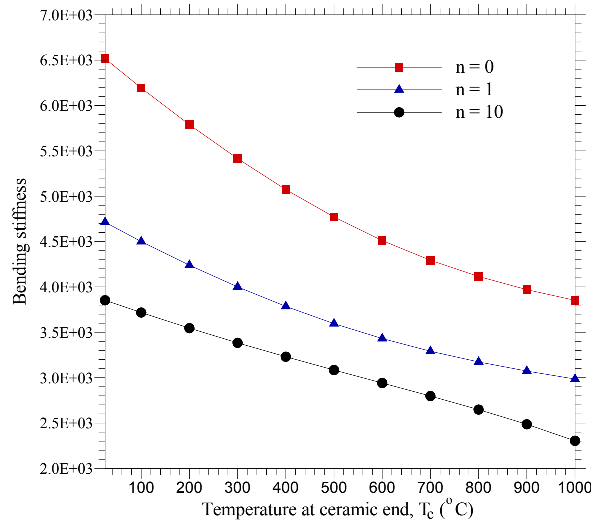
(d) M_{xx}^T

Fig. 8.7 Beam properties for FGM beam for different power-law index, n

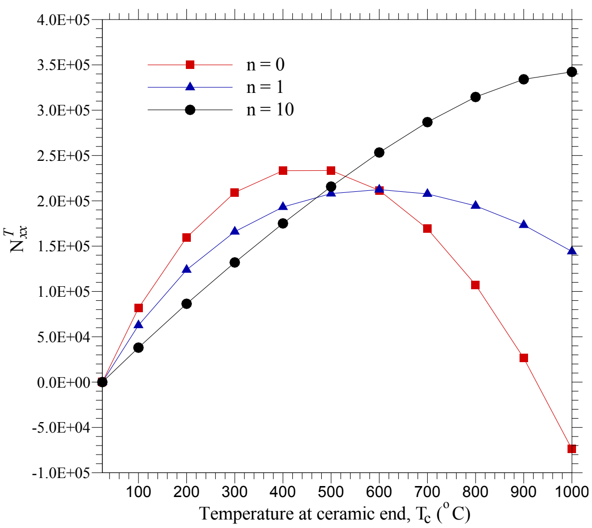
The variation of B_{xx}, bending stiffness $(D_{xx} - B_{xx}^2/A_{xx})$, N_{xx}^T and M_{xx}^T verses different temperature at the top (ceramic face), T_c for homogeneous and FGM beam having power-law index $n = 1$ and $n = 10$ are given in Fig. 8.8 considering temperature-dependent material properties.



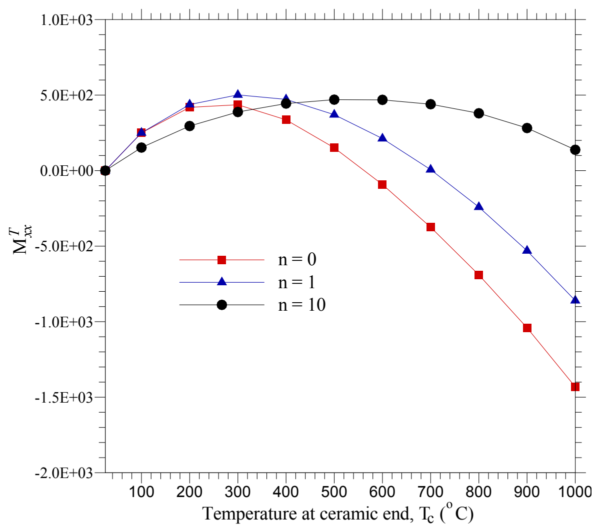
(a) B_{xx}



(b) Bending stiffness



(c) N_{xx}^T

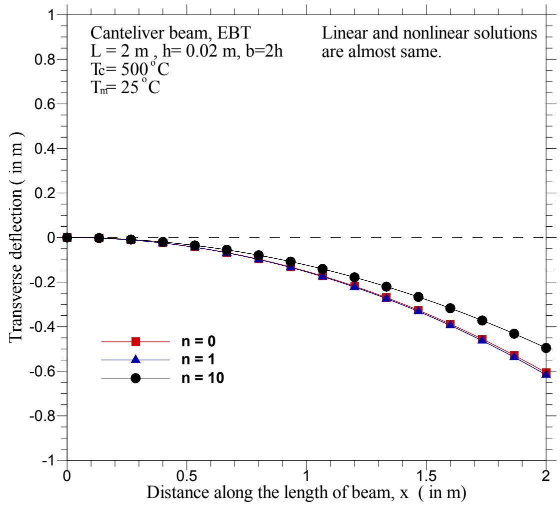


(d) M_{xx}^T

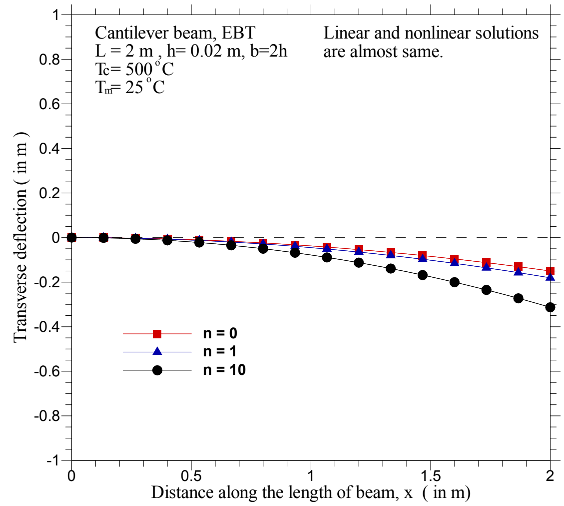
Fig. 8.8 Beam properties for FGM beam for different temperature at ceramic end, T_c

8.2.2. Cantilever beam

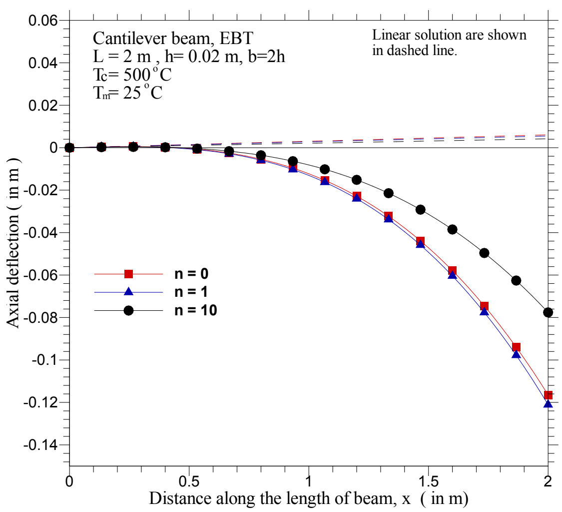
Nonlinear finite element (FE) analysis is performed for the cantilever beam of the geometric dimensions stated in Eq. (8.3). Sixty beam elements are taken for the FE



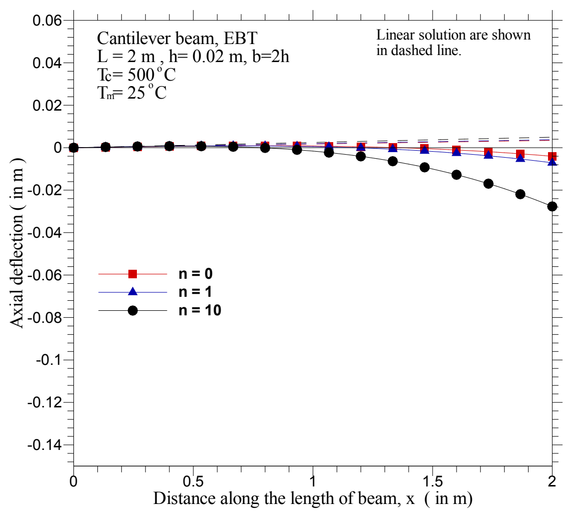
(a) for temperature independent material properties



(b) for temperature dependent material properties



(c) for temperature independent material properties

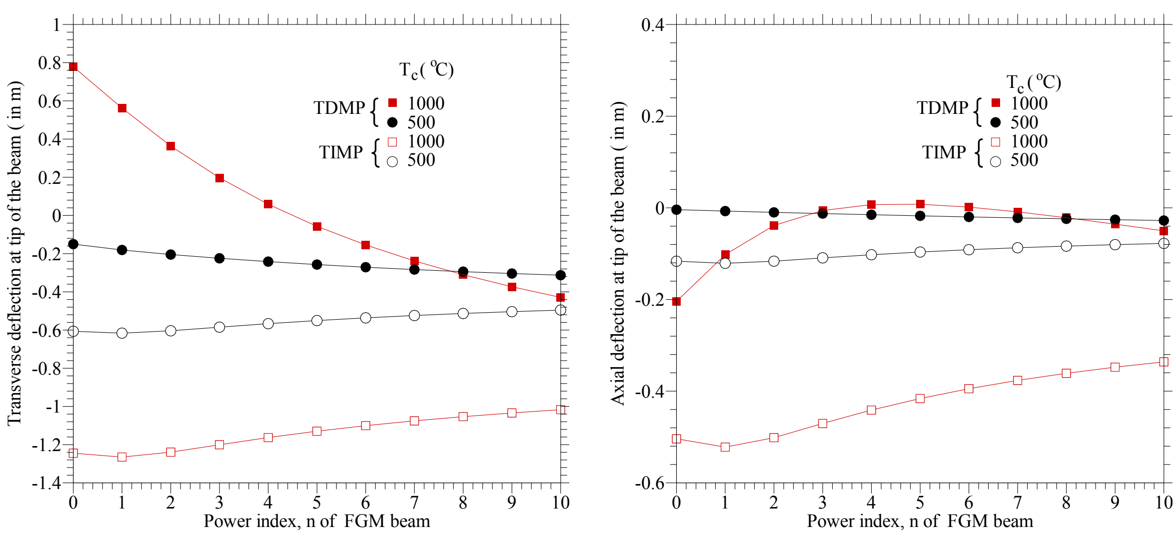


(d) for temperature dependent material properties

Fig. 8.9 Transverse and axial deflection of beam for different n of FGM for thermal load analysis of the beam considering Euler-Bernoulli beam theory where, axial displacement u and transverse displacement w are interpolated by linear and hermite cubic polynomial function respectively. The boundary condition for cantilever beam is as follows:

$$u(0) = 0, \quad w(0) = 0, \quad \theta(0) = 0 \tag{8.4}$$

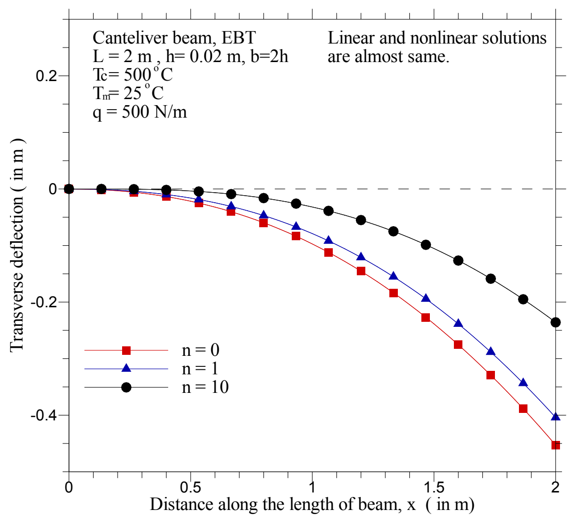
Analysis has been done for both temperature-dependent and temperature-independent material properties to compare the results for transverse and axial defections.Transverse and axial deflections along the length of the beam are shown in Fig. 8.9 for $T_c = 500°C$ for homogeneous and FGM beam having power-law index $n = 1$ and $n = 10$ considering only thermal load. The transverse and axial displacement of the tip of the cantilever beam have been plotted verses the power-law index of FGM beam for $T_c = 500°C$ and $T_c = 1000°C$ in Fig. 8.10. Plots are compared for both temperature dependent and independent material properties and it is noted that the difference between the two results are significant for high value of T_c.



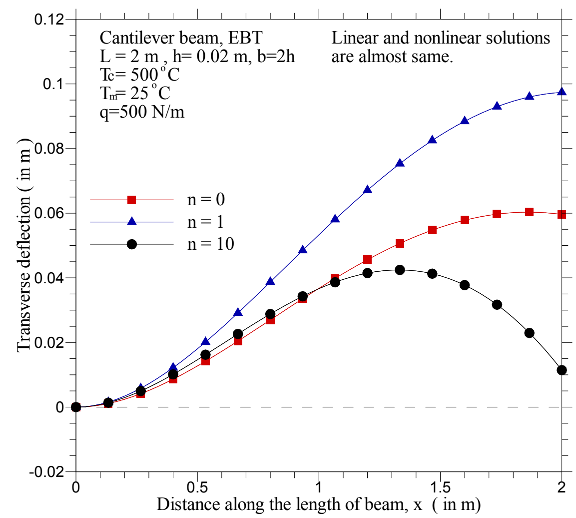
(a) Transverse deflection (b) Axial deflection

Fig. 8.10 Transverse and axial displacement of tip of the cantilever beam versus power-law index, n

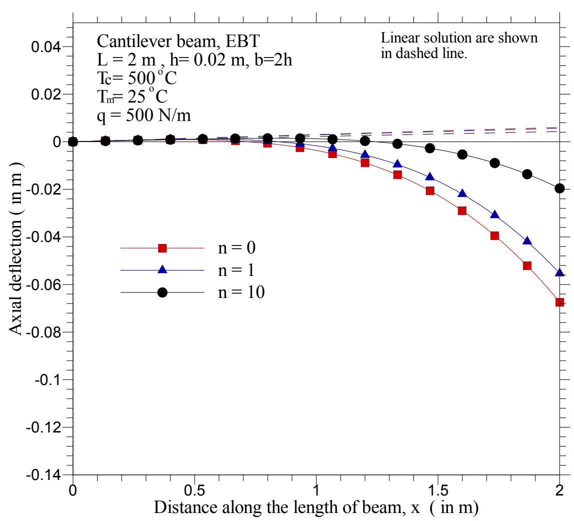
In Fig. 8.11, transverse and axial displacement of the cantilever beam are plotted along the length of the beam for homogeneous beam and FGM beam for thermal load along with uniformly distributed mechanical load of q = 500 N/m along the positive z-direction. The thermal boundary condition is taken as $T_c = 500°$C at the ceramic face and $T_m = 25°$C at the metal face.



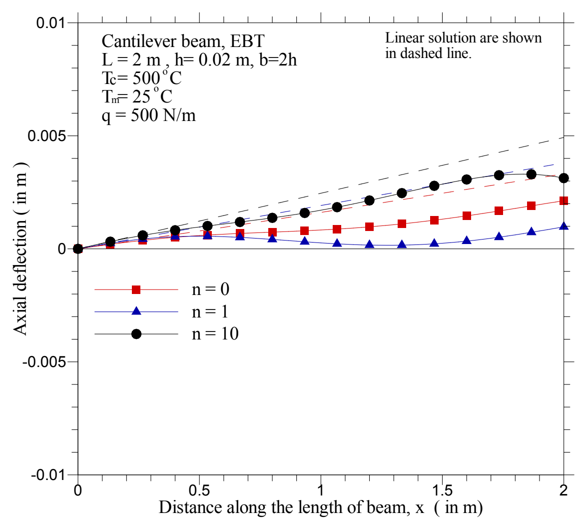
(a) for temperature independent material properties



(b) for temperature dependent material properties

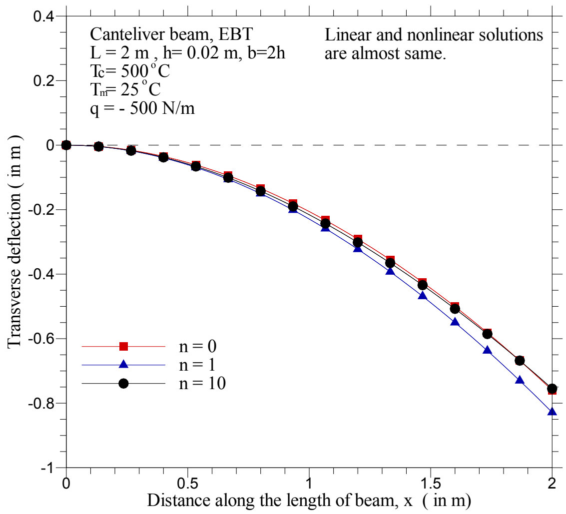


(c) for temperature independent material properties

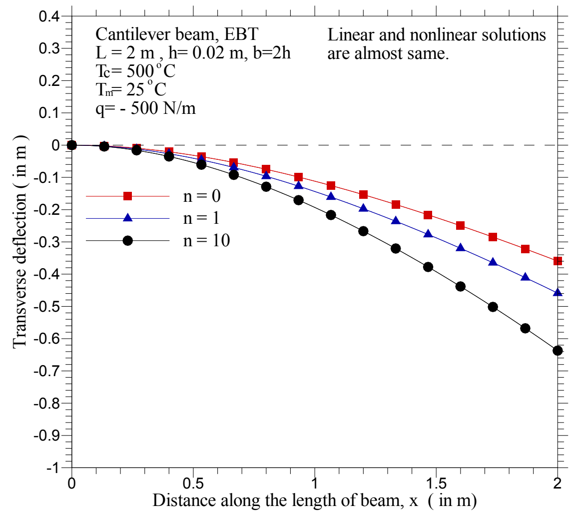


(d) for temperature dependent material properties

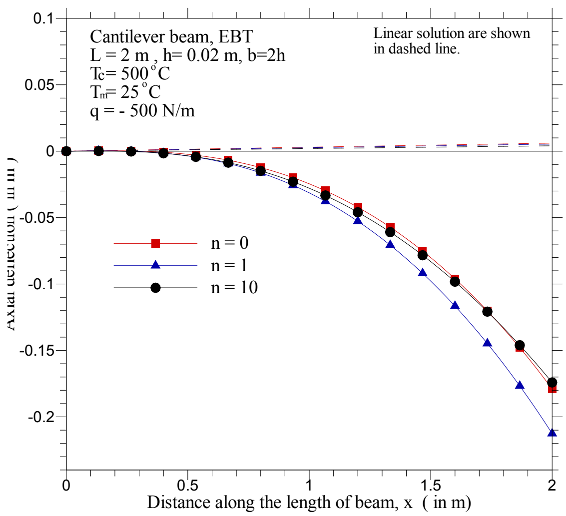
Fig. 8.11 Transverse and axial deflection of a beam for different n of FGM for thermo-mechanical load, q = 500 N/m



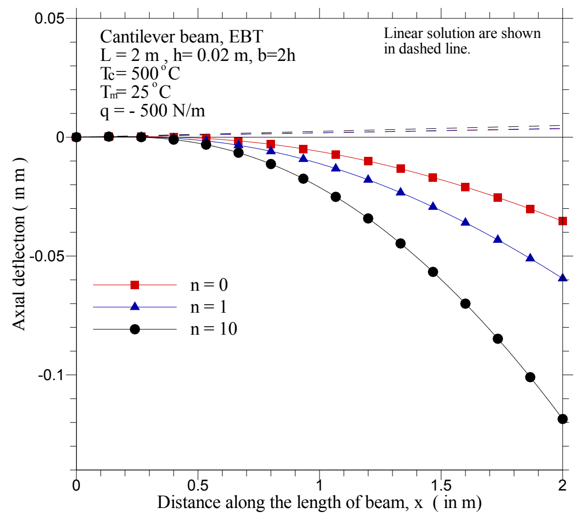
(a) for temperature independent material properties



(b) for temperature dependent material properties



(c) for temperature independent material properties

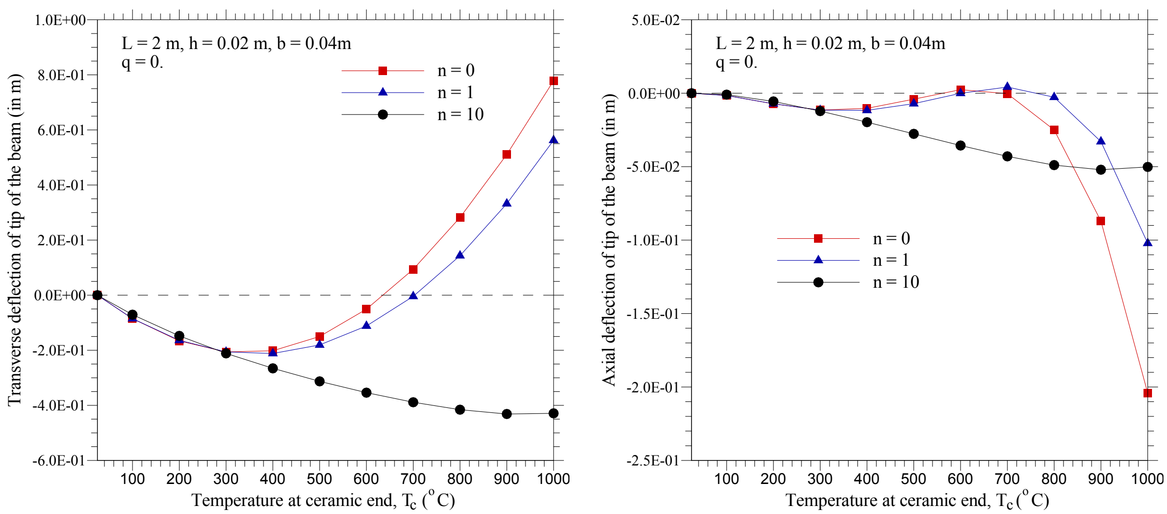


(d) for temperature dependent material properties

Fig. 8.12 Transverse and axial deflection of a beam for different n of FGM for thermo-mechanical load, $q = -500$ N/m

Similar plots for uniformly distributed mechanical loading $q = -500$ N/m along the negative z-direction are shown in Fig. 8.12. In Fig. 8.13, the transverse and axial displacement of the tip of the cantilever beam, i.e., the maximum displacement under the thermal load verses different boundary temperature at the ceramic end of the FGM beam have been plotted for homogeneous and FGM beam having power-law index $n = 1$ and $n = 10$. Here the displacement results are based on temperature dependent material properties.

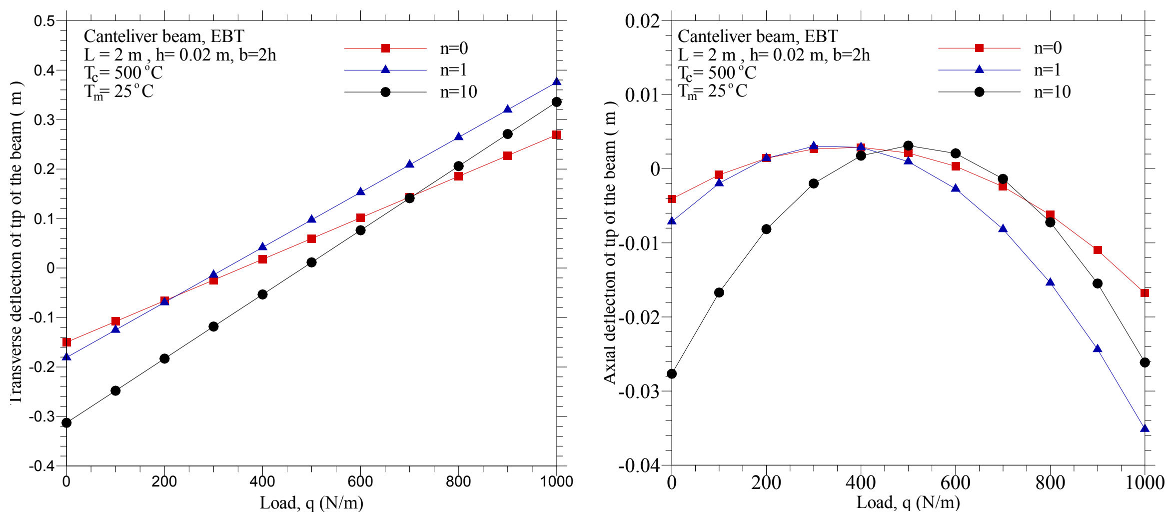
Transverse and axial displacement at the tip of the cantilever beams for thermo-mechanical load verses uniformly distributed mechanical loading, q_0 in the positive z-direction, are plotted for $T_c = 500°$ for homogeneous beam as well as for FGM beam having power-law index $n = 1$ and $n = 10$ in Fig. 8.14. Transverse displacement is almost linear for the cantilever beam, where as nonlinearity can be seen in the axial displacement of the beam.



(a) Transverse deflection

(b) Axial deflection

Fig. 8.13 Transverse and axial displacement of tip of the cantilever beam versus temperature at ceramic face, T_c



(a) Transverse deflection (b) Axial deflection

Fig. 8.14 Transverse and axial displacement of tip of the cantilever beam versus mechanical load applied

8.2.3. Pinned-pinned connected beam

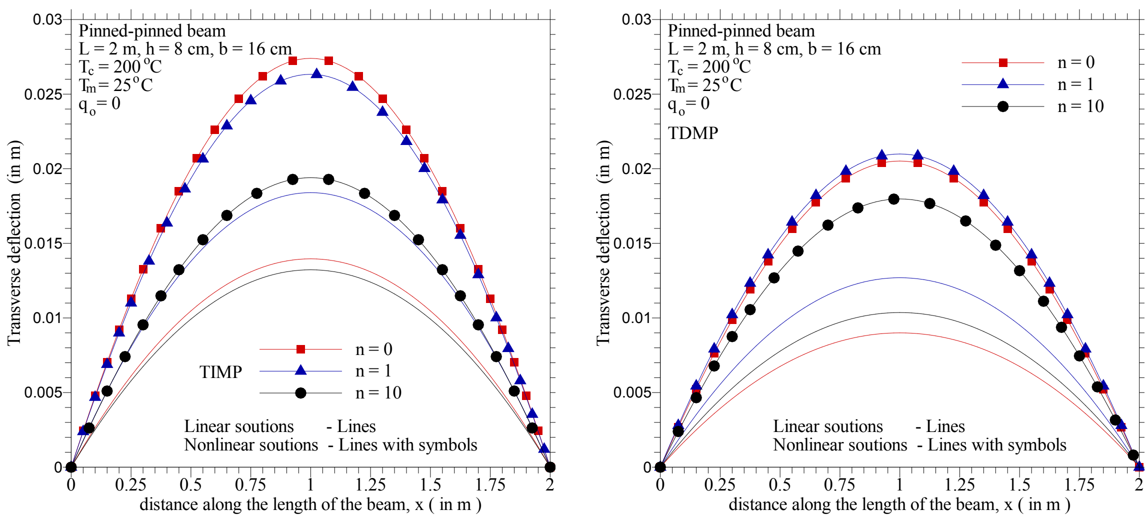
Pinned-Pinned beam of the following dimensions is taken for the present nonlinear analysis:

$$L = 2 \text{ m}, \quad \frac{L}{h} = 25, \quad h = 0.08 \text{ m}, \quad b = 2h = 0.16 \text{ m} \tag{8.5}$$

For non-linear finite element analysis, forty finite element have been used for the half domain of the pinned-pinned connected beam. The axial and transverse displacement of the nodes are interpolated using linear and hermite cubic interpolation function respectively for Euler-Bernoulli beam.Only half domain of the beam is considered because of the symmetry of the problem. The boundary condition for the half domain can be given as:

$$u(0) = 0, \quad w(0) = 0, \quad and \quad \theta(L) = 0, \quad u(L) = 0 \tag{8.6}$$

The temperature at the top ceramic face is taken as 200°C, whereas the bottom metal face is considered at room temperature 25°C. In case of pinned-pinned connection, at higher T_c, the beam may undergo buckling. Fig. 8.15 shows the transverse deflection along the length of the beam for temperature independent and temperature dependent material properties for homogeneous and FGM beam with power-law index $n = 1, 10$ for thermal load only. Both linear and nonlinear solutions are plotted. Maximum deflection of the beam verses different power-law index n of FGM beam is plotted in Fig. 8.16. Furthermore, the analysis has been performed for mechanical



(a) for temperature independent material properties

(b) for temperature dependent material properties

Fig. 8.15 Transverse deflection of a pinned-pinned FGM beam for thermal load

load along with thermal load. Fig. 8.17 shows the deflection of the beam for uniform distributed load in the positive z direction whereas, Fig. 8.18 shows the same for uniform distributed load in the negative z direction for homogeneous and FGM beam with power-law index $n = 1, 10$ considering temperature dependent and independent material properties. To see the nonlinearity, maximum deflection verses the temperature at ceramic end T_C and verses different values of uniformly distributed load applied have been plotted in Figs. 8.19 and 8.20, respectively.

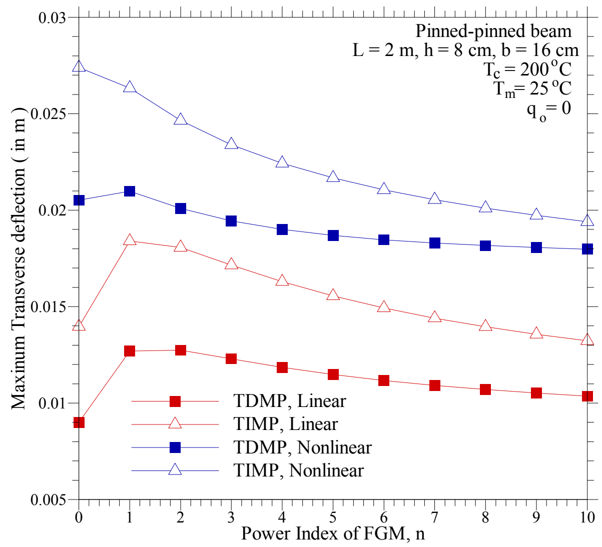
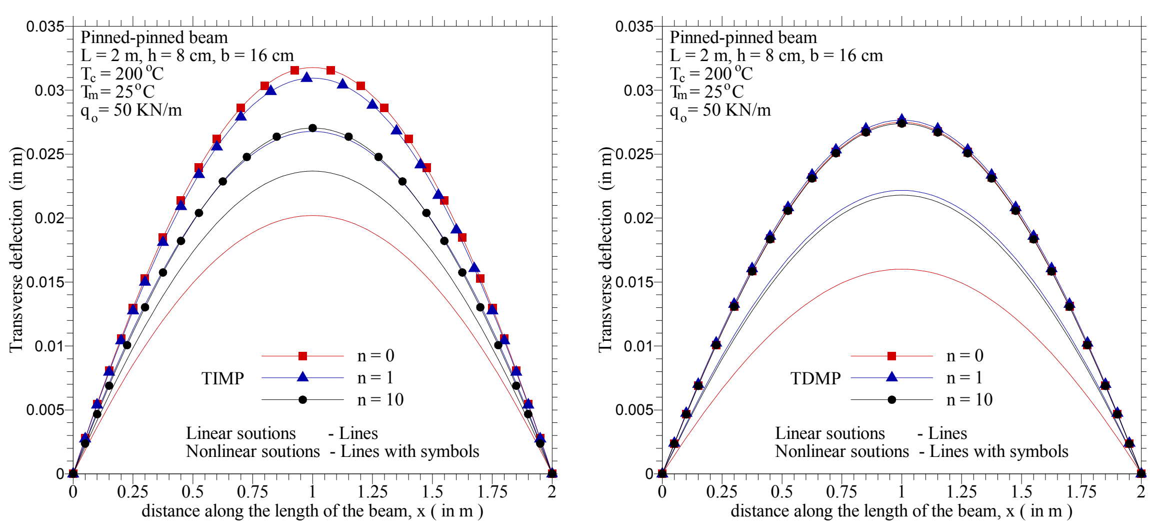


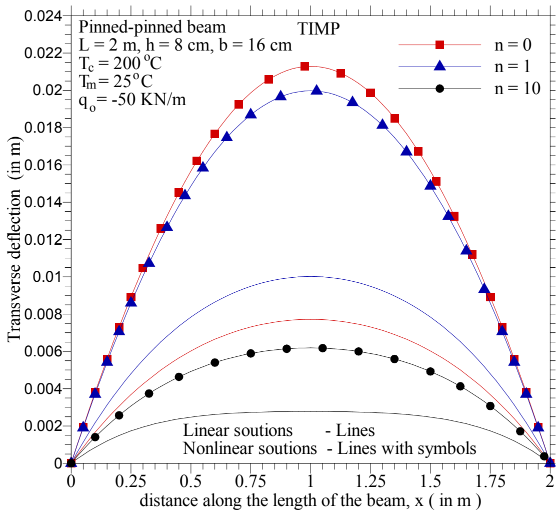
Fig. 8.16 Maximum transverse deflection versus power-law index, n.



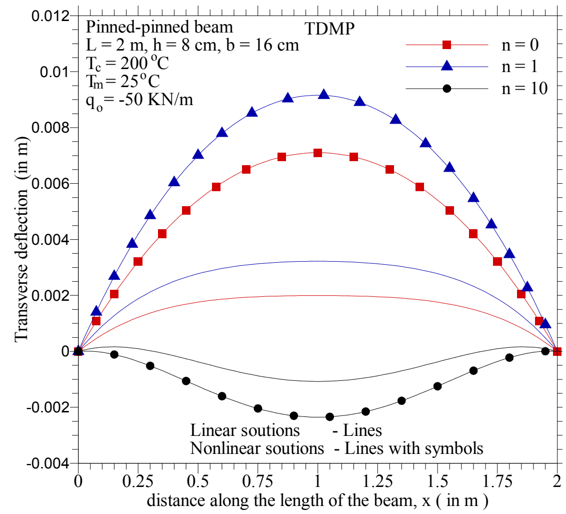
(a) for temperature independent material properties

(b) for temperature dependent material properties

Fig. 8.17 Transverse deflection of a pinned-pinned FGM beam for thermo-mechanical load $q = 50$ KN/m

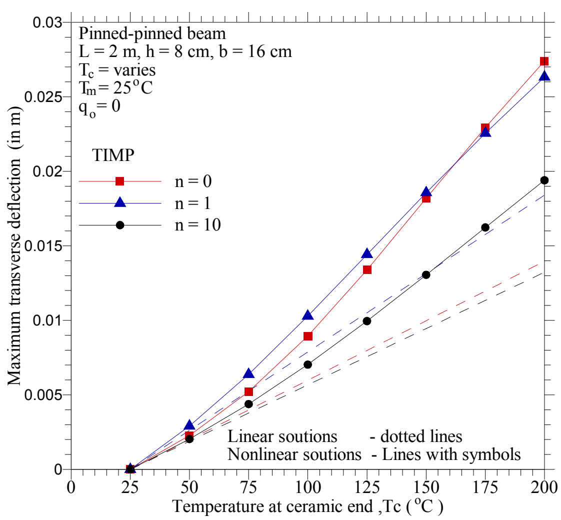


(a) for temperature independent material properties

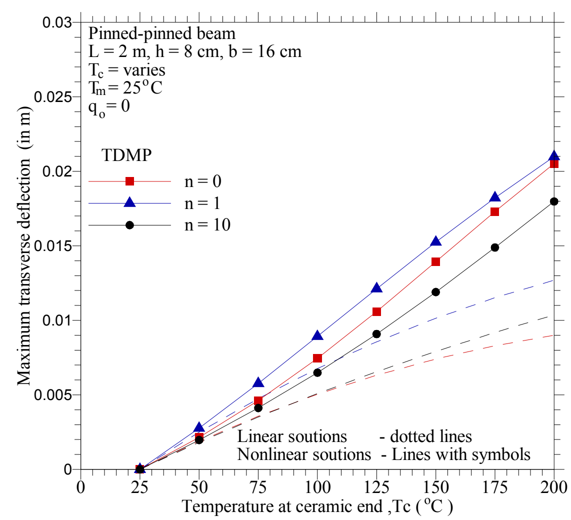


(b) for temperature dependent material properties

Fig. 8.18 Transverse deflection of a pinned-pinned FGM beam for thermo-mechanical load $q = -50$ KN/m

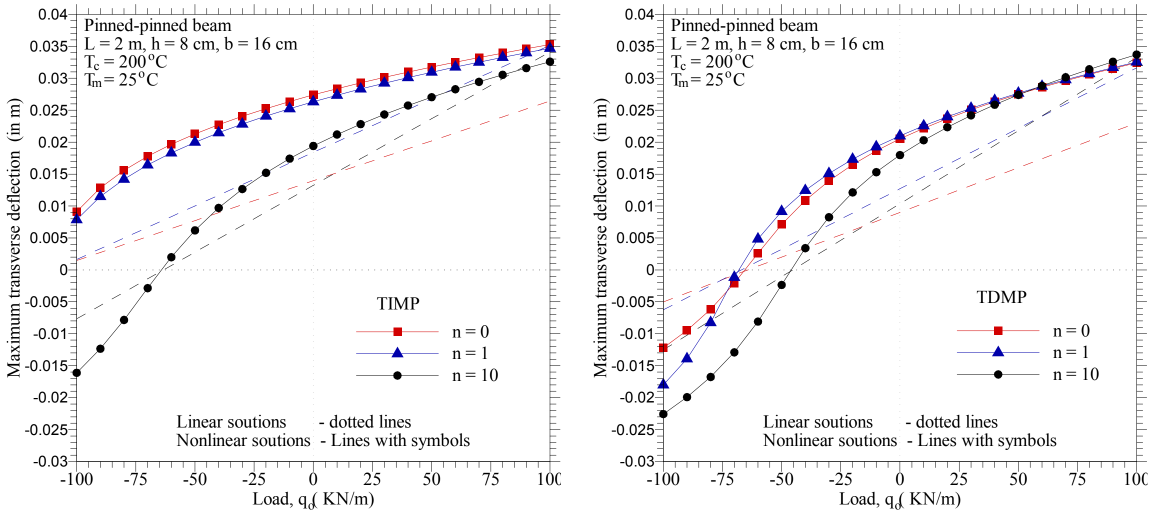


(a) For temperature independent material properties



(b) For temperature dependent material properties

Fig. 8.19 Maximum deflection of a pinned-pinned FGM beam versus temperature at ceramic end



(a) For temperature independent material properties

(b) For temperature dependent material properties

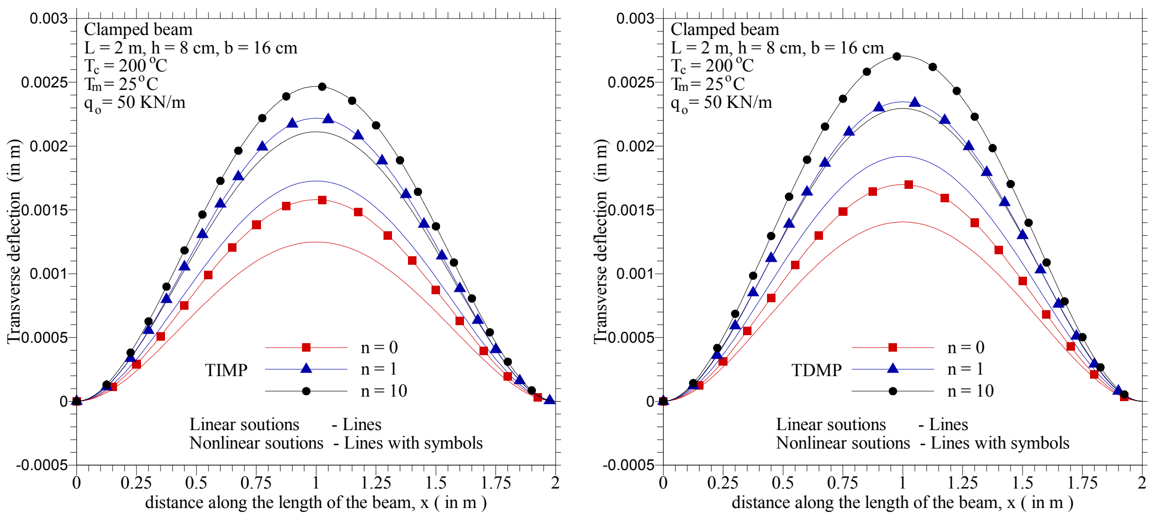
Fig. 8.20 Maximum deflection of a pinned-pinned FGM beam versus load applied

8.2.4. Clamped beam

The beam with geometric specification of Eq. (8.5) is analyzed for fixed (clamped) boundary condition. Again the half domain is considered for nonlinear finite element analysis because of the symmetry of the problem. Forty beam elements are taken for the half domain for the analysis. The boundary condition for half domain can be given as:

$$u(0) = 0, \quad w(0) = 0, \quad \theta_x = 0, \quad and \quad \theta(L) = 0, \quad u(L) = 0 \tag{8.7}$$

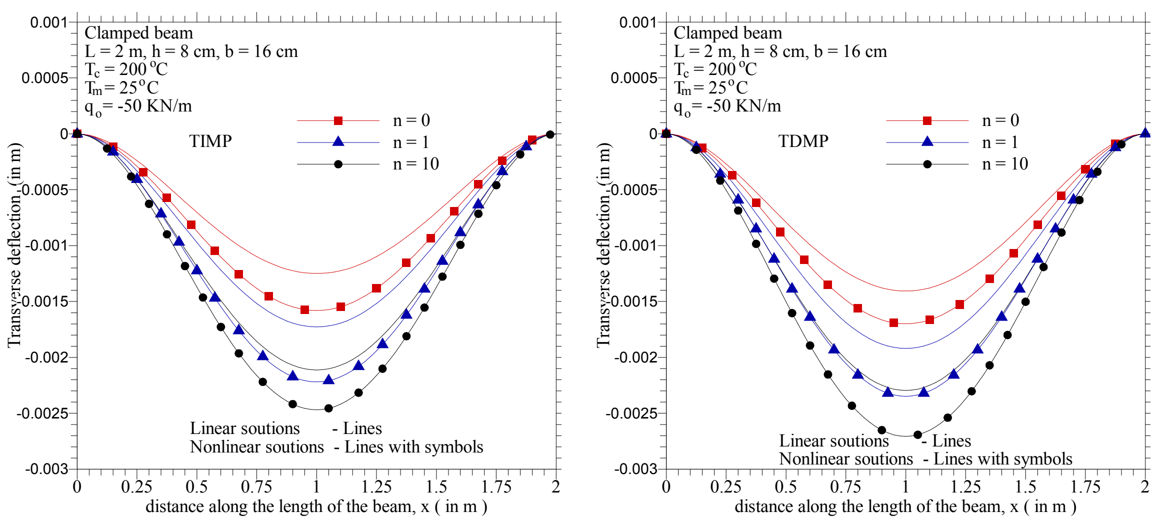
The temperature at the ceramic face (top) of the beam, (T_c) is again taken as 200°C and the same at metal fece is considered as room temperature (25°C). Only for thermal loading, there would not be any deflection as the ends are fully constrained. Figs. 8.21 and 8.22 show the transverse deflection along the length of the beam for uniformly distributed load in positive and negative z direction respectively for homogeneous and FGM beam having power-law index, $n = 1, 10$ considering temperature independent and temperature dependent material properties. Both linear and nonlinear solutions are plotted. In Fig. 8.23 maximum deflection versus mechanical load applied are plotted.



(a) Maximum deflection versus T_C

(b) Maximum deflection versus load

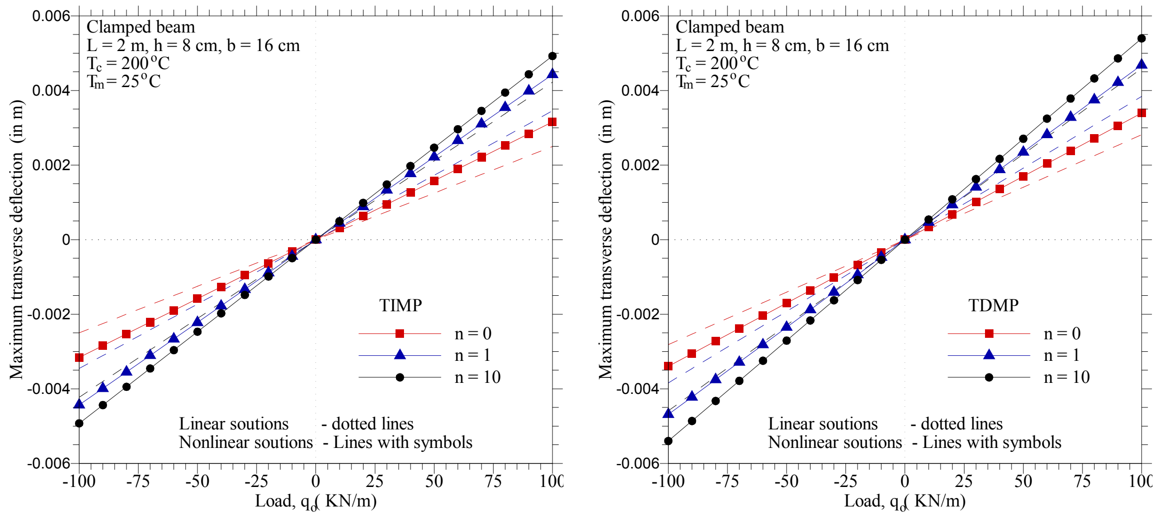
Fig. 8.21 Transverse deflection of a clamped FGM beam for thermo-mechanical load $q_0 = 50$ KN/m



(a) Maximum deflection versus T_C

(b) Maximum deflection versus load

Fig. 8.22 Transverse deflection of clamped FGM beam for thermo-mechanical load $q_0 = -50$ KN/m



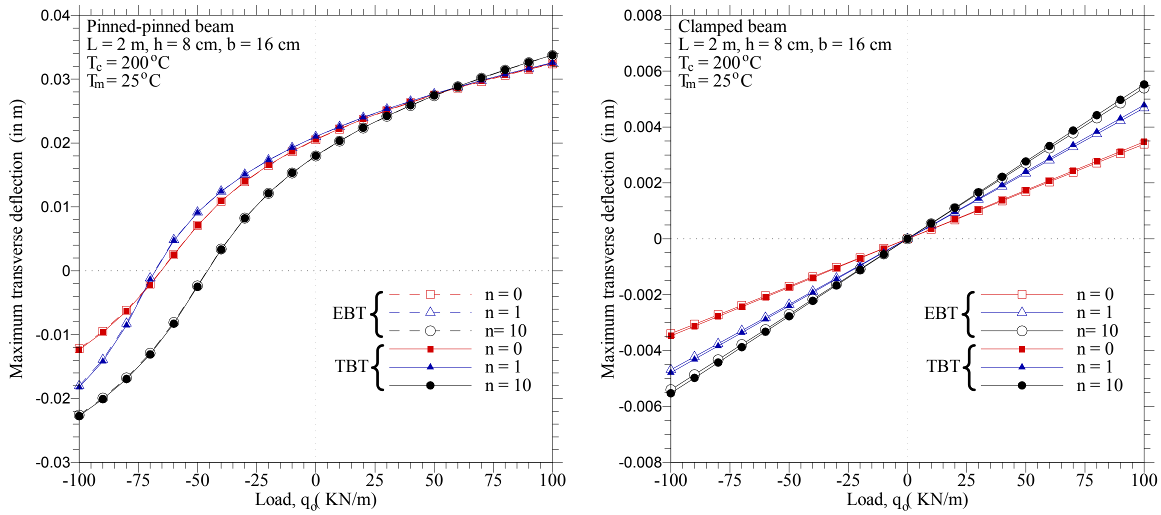
(a) Temperature independent material properties

(b) Temperature dependent material properties

Fig. 8.23 Maximum transverse deflection versus load applied for clamped FGM beam

8.2.5. Shear effect

To see the shear effect, numerical results considering Euler-Bernoulli beam theory and Timoshenko beam theory are compared for pinned-pinned and clamped boundary condition. Fig. 8.24 shows the maximum deflection of the beam considering both the theories for pinned-pinned and clamped beam of geometric specification given in Eq. (8.5). For $L/h = 25$, as in Eq. (8.5), the shear effect is not so significant as the difference between the EBT and TBT results are very small. It is also notable that clamped beam has more shear effect than the pinned-pinned connected beam. Figs. 8.25 and 8.26 shows the maximum deflection, ($\bar{w} = w_{max} \times (D_{xx} - B_{xx}^2/A_{xx})/100L^4$) with respect to the load applied on clamped beam for $L/h = 25$ and $L/h = 10$ for homogeneous and FGM beam respectively. Here, for the analyses, temperature dependent material properties of the beam is considered. It is clear from the figures that for $L/h = 10$, there is more shear effect.



(a) Temperature independent material properties

(b) Temperature dependent material properties

Fig. 8.24 Comparison of EBT and TBT solution for maximum transverse deflection versus load applied for clamped FGM beam

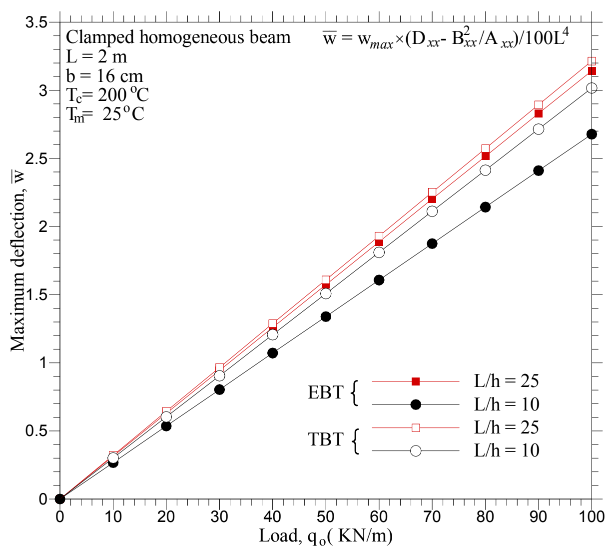
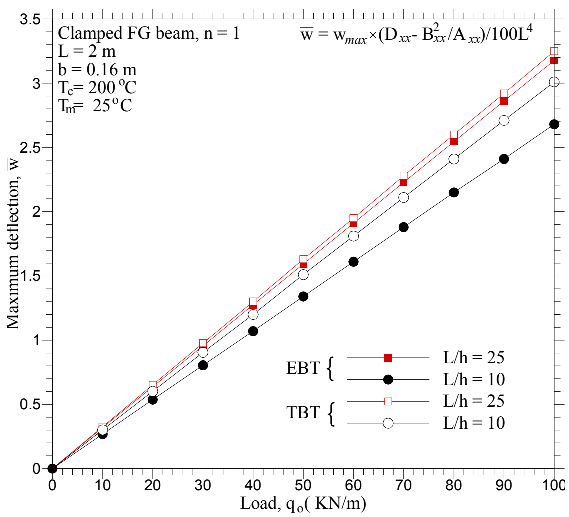
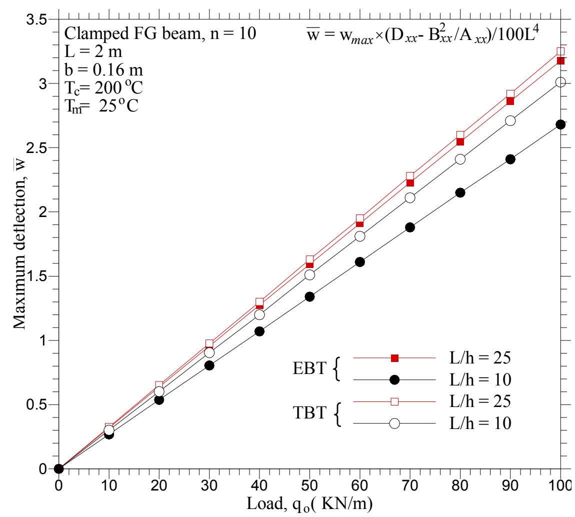


Fig. 8.25 Maximum deflection versus load applied for homogeneous clamped beam for different L/h ratio.



(a) FGM beam, $n = 1$



(b) FGM beam, $n = 10$

Fig. 8.26 Maximum transverse deflection versus load applied for clamped FGM beam for different L/h ratio

9. SUMMARY AND CONCLUSIONS

In this study nonlinear finite element analysis of functionally graded material beams without and with microstructural length scale under thermo-mechanical loads is carried out. Finite element models of the Euler–Bernoulli, Timoshneko, and third-order beam theories are developed. In case of microstructure dependent beam, the governing equations are based on modified couple stress theory. Apart from Euler-Bernoulli and Timoshenko beam theory, the governing equations of motion are also formulated for a general third order beam theory, which is again based on modified couple stress theory, the von Kármán nonlinearity, and functionally graded material beam. An analytical solution for pinned-pinned connected beam has been obtained in the linear case for the general third-order beam theory.The nonlinear FE model has been created for three beam theories namely Euler Bernoulli beam theory, Timoshenko beam theory and a general third order beam theory. To obtain the numerical solutions for various boundary conditions, finite element program has been developed in MATLAB.

In the numerical result section, First results has been discuss for microstructure dependent beam for homogeneous beam and functionally graded beam and the effect of microstructure length scale factor (ℓ) on the deflection of the beam has been seen. Stiffness of the beam increases for higher value of ℓ/h ratio. It can also be seen from the analysis that shear effect is prominent in case of higher (ℓ/h) ratio.

Further, the effect of thermal load in case of FGM beam have been investigated. Results for both temperature dependent and temperature independent material properties have been presented to see the effect of temperature dependent material properties. It is apparent from the results presented that for large thermal loads (i.e., high temperature conditions), dependance of material properties on temperature plays a significant role in the response. The response is influenced by (1) the nonlinearity associated with the thermal load, (2) the coupling stiffness B_{xx}, and (3) the von Kármán strain. The effect of shear deformation can be prominently seen for the clamped boundary condition and comparatively short beams.

In the postprocessing of finite element results, stresses in the beam can be calculated, from the design prospective, at desired points of the beam. Inclusion of material nonlinearity and damage in the FGM beam models is a good further study that might bring more understanding about the overall behavior of FGM beams under thermo-mechanical conditions. Such information would help in the design of these materials for use in micro- and nano-devices.

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